PHYSICAL REVIEW VOLUME 135, NUMBER 4A 17 AUGUST 1964

Degenerate Bose System. II. Λ Transformation of Quantum Statistical Theory

FRANZ MOHLING*
Tata Institute of Fundamental Research, Bombay, India

(Received 27 December 1963)

In the first paper of this series, the master-graph formulation of the Lee-Yang quantum statistical theory for a degenerate Bose system was derived. In the present paper, this entire theory is transformed from a free-particle description to a quasiparticle description by means of a Λ transformation. This transformation leaves the master-graph formulation of the theory essentially unchanged, while at the same time introducing the quasiparticle energy-momentum relation and the quasiparticle interaction function into the theory. The transformation is motivated by a study of the very low-temperature behavior of the fundamental integral equation of the theory. An interesting feature of the transformed theory is that it contains two quasiparticle energy-momentum relations, $\epsilon_+(k)$ and $\epsilon_-(k)$. The understanding of this result is not achieved in this paper, although the possibility of a double quasiparticle spectrum has previously been suggested by Lieb.

1. INTRODUCTION

IN the first paper of this series,[1] we have developed a quantum statistical theory of the degenerate Bose system, extending the earlier work of Lee and Yang.[2] In our first paper, particular attention was devoted to the self-energy structure of the graphs of the theory, and a final master-graph prescription was developed for the various physical quantities, in which the line factors included the sum over all possible self-energy structures In particular, both the grand potential and the momentum distribution were written down in the master-graph formulation of quantum statistics.

In the present paper the formal analysis is continued and concluded by the Λ transformation of the entire theory from a free-particle description to a quasiparticle description. The reader will undoubtedly wonder at the tremendous detail which has been included in a formal way in this paper. The explanation for this detail is twofold. In the first place, it seems to be mathematically necessary to go through all of the steps of this paper in order to arrive at a transformed theory, with which the application to a real or model degenerate Bose system may be relatively straightforward. Thus, it is extremely likely that the results of this paper can be applied to the study of the microscopic theory of liquid helium II, throughout its temperature range, by a simple perturbation or graphical series expansion of the general expressions. If so, then this will be a tremendous advantage of the theory. In the second place, this detail has been a consequence of the studies of the model Bose system of a dilute gas of hard spheres. Continued attempts to arrive at the well-known Lee-Huang-Yang expressions for the ground-state properties of this system[3] have uncovered the many subtleties of the theory which are presented here. Without their results, the research which has led to this paper might easily have failed to uncover the most important features of the quasiparticle theory.

What does one mean by a quasiparticle theory? The interpretation which we give to a quasiparticle theory is that it is a theory in which the quantum mechanical normal modes of a system are exhibited. Thus, although real systems consist of real interacting particles, nature allows a description of real systems in terms of interacting quasiparticles for which the interactions are minimized. Of course, it would be nice if nature would allow a description in terms of free quasiparticles, but this seems never to be the case. For example, the quasiparticles in a crystal, called phonons in this case, can only be considered to be free to first approximation. Similarly, in the microscopic theory of nuclear matter,[4] one can deduce a quasiparticle description which is in agreement with the macroscopic Landau theory of a Fermi liquid, and in this case one also finds that the quasiparticles interact. If one has once (theoretically) discovered the quasiparticle description of a system, then the physical properties of the system can be calculated by using an appropriate perturbation theory applied to the quasiparticle (or residual) interactions. This is the advantage of a quasiparticle description.

In the present paper we do not *exhibit* a quasiparticle model for a degenerate Bose system, yet we nevertheless believe that we have arrived at the quasiparticle description. The reason for this belief is that we feel that the quasiparticle description must be intimately associated with a correct treatment of the momentum space ordering in the degenerate Bose system, and this latter problem is the one we have considered in detail here. Thus, we consider a degenerate Bose system at rest, for which the occupation of the zero-momentum state is macroscopic. This macroscopic occupation is characterized by a nonzero value for the density $\langle x\rangle$ of zero-

* United States Government Fulbright Research Scholar in India, on leave from the Physics Department, University of Colorado, Boulder.

[1] F. Mohling, Phys. Rev. **135**, A831 (1964). Hereafter referred to as MI.

[2] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 897 (1960).

[3] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957). See also T. D. Lee and C. N. Yang, *ibid.* **112**, 1419 (1958).

[4] F. Mohling, in *Lectures in Theoretical Physics*, edited by W. Brittin, B. Downs, and J. Downs (Interscience Publishers, Inc., New York, 1962), Vol. IV, p. 436.

momentum particles. One can calculate $\langle x\rangle$ in this theory, as shown in MI, and if one finds that $\langle x\rangle=0$, then the theory reduces to that of a "normal" system.

In the case that $\langle x\rangle\neq 0$, and this is the case for liquid helium II, we find in this paper a quantity $-\Delta^{(0)}$ which we identify as the energy per particle of the "superfluid" zero-momentum particles. Since, for the superfluid, the energy is the same thing as the free energy, we set this quantity equal to the chemical potential g [Eq. (115)], a step which we can mathematically justify on an *a priori* basis only at $T=0$. Similarly, the study of the momentum space ordering of the nonzero-momentum particles in the system leads to a self-energy $-\Delta_{\pm}(\mathbf{k})$, Eq. (26), which we can identify as the potential energy of a quasiparticle with momentum $\mathbf{k}$. The energy of this quasiparticle, relative to the superfluid, is then given by $\epsilon_{\pm}(\mathbf{k})=\omega(\mathbf{k})-\Delta_{\pm}(\mathbf{k})+\Delta^{(0)}$, where $\omega(\mathbf{k})$ is the free-particle energy [see Eqs. (40) and (131)]. Now, the peculiar thing is that we have obtained *two* quasiparticle energies $\epsilon_{+}(\mathbf{k})$ and $\epsilon_{-}(\mathbf{k})$, and this is a result for which we do not yet have an interpretation. It is interesting to note, however, that the suggestion that a double quasiparticle spectrum might exist in a degenerate Bose system has previously been made by Lieb.[5] Finally, we have also arrived at the quasiparticle interaction function, and this is the transformed pair function of Sec. 4.

It is very convenient to think in terms of the above quasiparticle interpretation when proceeding through the formal analysis of this paper. We have avoided the use of this language in writing the formal analysis, however, because the analysis is mathematically motivated with the interpretation coming afterwards. Therefore, we now proceed with a discussion of how the momentum space ordering is achieved from the mathematical point of view.

In Sec. 2 we itemize the basic quantities used in the master graphs of MI. In this way, we are able to emphasize those relations which form the basis of the subsequent analysis. In particular, we observe that the kernal $P(s,t,\mathbf{k})$ in the most important integral Eq. (12) of the theory has a part $P_0(s,t,\mathbf{k})$, which does not permit an iterative solution of (12) at very low temperatures. Thus, one is forced to solve the integral Eq. (18), and an exact solution of this equation is given in Sec. 3.

It should be remarked here that the integral Eq. (18) provides the important stepping stone to the Λ transformation of Sec. 5. Now, when the Λ transformation was first studied for the case $\langle x\rangle=0$, and for Fermi systems,[6] it was found that only the first two terms of $P_0(t_2,t_1,\mathbf{k})$, Eq. (16), entered into the theory. It is therefore quite natural to use this simplified form for $P_0(t_2,t_1,\mathbf{k})$ when $\langle x\rangle\neq 0$. Such an attempt fails completely, because the very low-temperature behavior of the theory is not properly treated when this is done This simplified approach also fails to yield the Lee-Huang-Yang results[3] for the hard-sphere Bose gas. One is thus *forced* to consider the entire complicated expression (16) for $P_0(t_2,t_1,\mathbf{k})$.

In Secs. 5–8, the Λ transformation of the theory is performed. This transformation starts from the observation that the solution $G_0(t_2,t_1,\mathbf{k})$ to the integral Eq. (18) is really only a first approximation to the solution to the basic Eq. (12). Therefore, the function $G_0(t_2,t_1,\mathbf{k})$ can occur along any of the internal lines of the master graphs as part of the self-energy factors, and therefore one encounters integrals over pair functions (the vertex functions of the theory) everywhere. These are the integrals (35) which are studied in detail in Sec. 4. In simplest terms, the Λ transformation is nothing more than the elimination of the explicit appearance of the (large) function $G_0(t_2,t_1,\mathbf{k})$ in the theory by performing the integrals (35). It is only when one wants to *insure* that these integrals are actually performed everywhere that one arrives at the concept of a linear integral transformation (on a very nonlinear theory). The full study of this Λ transformation, of which there are four different cases, is the content of Secs. 5–8. Thus, the basic Λ transformation is introduced in Sec. 5. In Sec. 6, the transformation of the line factors of the master graphs is studied, and an expression for the momentum distribution in the transformed theory is derived. In Sec. 7, the zero-momentum factors are transformed, and in Sec. 8, the grand potential is transformed. Finally, in Sec. 9, the four functions $\Lambda(t_2,t_1,\mathbf{k})$ are discussed in detail, for it is these functions which really characterize the Λ transformation equations. These functions and the related functions $\zeta(t,\mathbf{k})$ appear explicitly in the expressions for the transformed momentum distribution and grand potential.

From the above discussion one can see that the Λ transformation provides the solution to the low-temperature self-energy problem, which is presented by terms of the form $P_0(t_2,t_1,\mathbf{k})$. Thus, after the Λ transformation, the transformed basic integral equation (59) can be solved approximately by iteration (as can all the other transformed integral equations of the theory). The Λ transformation is therefore also a key to the microscopic understanding of the momentum space ordering in the degenerate Bose system, although this key can really only be turned by making an application of the theory to a real or model system. One is able to make a start towards understanding this ordering, however, by considering the few quantities which have already appeared in the formal analysis. Thus, the Λ transformation changes the free-particle energies (almost) everywhere to the functions $\omega(\mathbf{k})-\Delta_{+}(\mathbf{k})$, $\omega(\mathbf{k})-\Delta_{-}(\mathbf{k})$, and $-\Delta^{(0)}$, and this change strongly suggests the quasiparticle interpretation given above.

[5] Elliot H. Lieb and Werner Liniger, Phys. Rev. **130**, 1605 (1963); Elliot H. Lieb, *ibid.* **130**, 1616 (1963).

[6] F. Mohling, Phys. Rev. **122**, 1062 (1961).

2. INVESTIGATION OF THEORY AT VERY LOW TEMPERATURES

In the development of a quantum statistical theory of the degenerate Bose system in MI, our principal objective was the analysis of the self-energy graphs of the theory. Thus, in the final formulation derived in MI, the grand potential and the distribution functions for an arbitrary Bose system are expressed in terms of master graphs whose line factors represent the sum over all possible self-energy graphs. We shall not repeat the rules and equations of this final prescription here; but rather, we shall summarize the ingredients, or building blocks, for master graphs. It is the investigation of these basic quantities which provides the motivation for the analysis of the present paper.

The first quantity which we shall write down is the pair function [MI, Eqs. (18)–(20)], which is the vertex function of the theory. The dynamics of the Bose system is determined by the pair function because it is the only function of the theory which explicitly depends on the elementary two-body interaction between two free bosons. In a preceding paper,[7] we have studied the pair function in detail. Thus, its general definition is

$$
{}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} = {}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_2-t_1)\theta(t_1-t_0)+{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_1-t_2)\theta(t_2-t_0) \quad \text{if} \quad t_1\neq t_2\,,
$$
$$
= {}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_1-t_0) \qquad \text{if} \quad t_1=t_2\,, \tag{1}
$$

$$
{}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} = \langle\mathbf{k}_1\mathbf{k}_2|R(t_1,t_0)|\mathbf{k}_3\mathbf{k}_4\rangle+\langle\mathbf{k}_1\mathbf{k}_2|R(t_1,t_0)|\mathbf{k}_4\mathbf{k}_3\rangle\,, \tag{2}
$$

where

$$
\begin{cases} R(t_2,t_1)=-W_2(t_2,t_1)V(t_1)\,, \\ W_2(t_2,t_1)=1-\displaystyle\int_{t_1}^{t_2} ds W_2(t_2,s)V(s)\,, \end{cases} \tag{3}
$$

$$
V(t)=\exp(tH_0^{(2)})V_2\exp(-tH_0^{(2)})\,, \tag{4}
$$

$$
H^{(2)}=H_0^{(2)}+V_2\,. \tag{5}
$$

Equations (3) give the operator form of the pair function in terms of the two-body potential in the interaction representation (4). Equation (5) expresses the two-body Hamiltonian $H^{(2)}$ as a sum of a free-particle part $H_0^{(2)}$ and the two-body interaction V_2. The $\theta(y)$ in Eq. (1) are step functions, defined by $\theta(y)\equiv 1$ if $y>0$ and $\theta(y)\equiv 0$ if $y<0$. It can be seen by iterating the expression for $W_2(t_2,t_1)$ and then substituting the result into $R(t_2,t_1)$, that the pair function (2) is a sum over all "ladder diagrams" (using the language of field theory). For a weak potential, one can use simple perturbation theory to determine the pair function.

Most realistic two-particle interactions contain a repulsive core, and this is certainly true of the interaction between helium atoms. Since the repulsive core plays a dominant role in the Bose many-body problem,[8] one cannot use perturbation theory to determine the pair function. Now, although the two-body Schrödinger equation for a realistic helium interatomic potential cannot be solved exactly, one can nevertheless write down the form which Eq. (2) takes for a general potential by using techniques from scattering theory. The result[7] is

$$
\begin{aligned}
{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} &= \exp[t_1(\omega_1+\omega_2-\omega_3-\omega_4)]f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\sum_{\mathbf{k}_5\mathbf{k}_6}\exp[t_2(\omega_1+\omega_2-\omega_5-\omega_6)] \\
&\quad\times\exp[t_1(\omega_5+\omega_6-\omega_3-\omega_4)]f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)+\delta(t_2-t_1) \\
&\quad\times\exp[t_1(\omega_1+\omega_2-\omega_3-\omega_4)]f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\,,
\end{aligned} \tag{6}
$$

where each of the f_i functions can be expressed entirely in terms of two-particle reaction matrices, which are well defined, even for an infinite repulsive core interaction. The free-particle energies ω_i are given by $\omega(\mathbf{k})=\hbar^2k^2/2M$ in the limit of an infinite system, and the function f_3 is to be included only when one wishes to use the mathematical idealization of an infinite repulsive core. For a finite or penetrable repulsive core, $f_3\equiv 0$. The usefulness of Eq. (6) is that it explicitly exhibits the

[7] F. Mohling, Phys. Rev. **122**, 1043 (1961). See also F. Mohling, *ibid.* **124**, 583 (1961).

[8] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. II. On pp. 30–35, the importance of the hard core to a microscopic understanding of liquid helium II is demonstrated.

form of the temperature dependence of the pair function for an arbitrary two-particle interaction.

We next consider the line factors of master graphs. These are given as the solutions of integral equations, derived in MI. We shall first concentrate our attention on the kernels and inhomogeneous parts of these integral equations. As we have previously shown, the line factors arise from both dynamical and statistical effects. The effect of statistics, or exchange, is primarily determined by the function $\nu(\mathbf{p})$,

$$\nu(\mathbf{p})=\exp\beta(g-\omega_{\mathbf{p}})[1-\exp\beta(g-\omega_{\mathbf{p}})]^{-1}, \tag{7}$$

where g is the thermodynamic potential per particle in the system and $\beta=(\kappa T)^{-1}$. In this paper we continue to use the convention of MI that $\mathbf{k}\to\mathbf{p}$ when $\mathbf{k}$ cannot be zero, i.e., $\mathbf{p}\neq 0$. The function $\nu(\mathbf{p})$ occurs as an inhomogeneous term and as a factor in the kernels of the integral equations [MI, (35)–(37)] for the $N_{\mu,\nu}(\mathbf{p})$. The functions $N_{\mu,\nu}(\mathbf{p})$ then give the direct effect of statistics on the master-graph line factors $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$, Eqs. (66)–(72) in MI.

The dynamical part of the line factors is primarily due to the functions $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$ of Eq. (73) in MI. These functions given by

$$K_{\mu,\nu}(t_2,t_1,\mathbf{k})=\sum[\text{all different master }(\mu,\nu)\ L\text{ graphs}]_{\mathbf{k}}, \tag{8}$$

where $(\mu,\nu)=(1,1)$, $(0,2)$, or $(2,0)$, play a central role in the analysis of this paper. Closely related to the $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$ is the function [MI, Eq. (62)]

$$P(t_2,t_1,\mathbf{k})=K_{1,1}(t_2,t_1,\mathbf{k})+\int_0^\beta ds_1ds_2K_{2,0}(t_2,s_1,\mathbf{k})\times G(s_2,s_1,-\mathbf{k})K_{0,2}(t_1,s_2,\mathbf{k}), \tag{9}$$

where

$$G(t_2,t_1,\mathbf{k})=\delta(t_2-t_1)+L(t_2,t_1,\mathbf{k}), \tag{10}$$

$$L(t_2,t_1,\mathbf{k})=\int_0^\beta dsG(t_2,s,\mathbf{k})K_{1,1}(s,t_1,\mathbf{k}). \tag{11}$$

The functions $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$ and $P(t_2,t_1,\mathbf{k})$ are kernels in the integral equations [MI, (63)–(65)] for the $L_{\mu,\nu}(t_2,t_1,\mathbf{k})$, and the $L_{\mu,\nu}(t_2,t_1,\mathbf{k})$ then give the direct effect of dynamics on the line factors $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$. We shall require these integral equations in the present paper, and so we repeat them here.

$$L_{1,1}(t_2,t_1,\mathbf{k})=\int_0^\beta dsG_{1,1}(t_2,s,\mathbf{k})P(s,t_1,\mathbf{k}), \tag{12}$$

$$L_{0,2}(t_2,t_1,\mathbf{k})=\int_0^\beta ds_2ds_1K_{0,2}(s_2,s_1,\mathbf{k})G_{1,1}(s_2,t_2,\mathbf{k})\times G(s_1,t_1,-\mathbf{k})-K_{0,2}^{(1)}(t_2,t_1,\mathbf{k}), \tag{13}$$

$$L_{2,0}(t_2,t_1,\mathbf{k})=\int_0^\beta ds_2ds_1G_{1,1}(t_2,s_2,\mathbf{k})G(t_1,s_1,-\mathbf{k})\times K_{2,0}(s_2,s_1,\mathbf{k})-\delta(t_2,t_1)K_{2,0}^{(1)}(t_2,t_1,\mathbf{k}), \tag{14}$$

where

$$G_{1,1}(t_2,t_1,\mathbf{k})=\delta(t_2-t_1)+L_{1,1}(t_2,t_1,\mathbf{k}), \tag{15}$$

and the functions $K_{0,2}^{(1)}(t_2,t_1,\mathbf{k})$ and $K_{2,0}^{(1)}(t_2,t_1,\mathbf{k})$ are defined below Eq. (59) in MI.

Having completed a review of the basic quantities in the theory, we are now in a position to begin the analysis of this paper. Our first step will be to write down a special class of terms which occurs in the function $P(t_2,t_1,\mathbf{k})$ of Eq. (9), and we shall define the sum of these terms to be $P_0(t_2,t_1,\mathbf{k})$. The general form of $P_0(t_2,t_1,\mathbf{k})$ is then

$$\begin{aligned}[1+B(\mathbf{k})]P_0(t_2,t_1,\mathbf{k})&=[A(\mathbf{k})+B(\mathbf{k})\delta(t_2-t_1)+C(\mathbf{k})]\theta(t_2-t_1)\\&\quad+C(\mathbf{k})\exp[(t_2-t_1)D(\mathbf{k})]\theta(t_1-t_2)\\&\quad-C'(\mathbf{k})\exp[-t_1D(\mathbf{k})]+\exp[-\beta D(\mathbf{k})]\\&\quad\times[B'(\mathbf{k})+B''(\mathbf{k})\exp t_2D(\mathbf{k})]\delta(\beta-t_1),\end{aligned} \tag{16}$$

where each of the quantities A, B, C, etc., may have a dependence on β, although this has not been explicitly indicated in (16). The factor $[1+B(\mathbf{k})]$ on the left-hand side of Eq. (16) is introduced for convenience [see below Eq. (19)]. One can demonstrate the existence of each of the terms in Eq. (16) by a simple lowest order calculation of each of the functions $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$ which contribute to Eq. (9). Thus, one has only to use the one-vertex master (μ,ν) L graphs of Fig. 1, in connection with Eq. (8), to obtain the expressions

$$\begin{aligned}K_{1,1}(t_2,t_1,\mathbf{k})_0&=(x\Omega e^{\beta g})\int_0^\beta dsG_{\text{out}}(s)\,{}^{t_2s}\!\begin{bmatrix}\mathbf{k}&0\\\mathbf{k}&0\end{bmatrix}_{t_1}G_{\text{in}}(t_1)\\&\cong(x\Omega e^{\beta g})\,{}^{t_2}\!\begin{bmatrix}\mathbf{k}&0\\\mathbf{k}&0\end{bmatrix}_{t_1},\end{aligned} \tag{17a}$$

$$\begin{aligned}K_{0,2}(t_2,t_1,\mathbf{k})_0&=\tfrac{1}{2}(x\Omega e^{2\beta g})\int_0^\beta ds_2ds_1\\&\quad\times G_{\text{out}}(s_2)G_{\text{out}}(s_1)\,{}^{s_1s_2}\!\begin{bmatrix}0&0\\\mathbf{k}&-\mathbf{k}\end{bmatrix}_{t_1}\delta(t_2-t_1)\\&\cong\tfrac{1}{2}(x\Omega e^{2\beta g})\,{}^{\beta}\!\begin{bmatrix}0&0\\\mathbf{k}&-\mathbf{k}\end{bmatrix}_{t_1}\delta(t_2-t_1),\end{aligned} \tag{17b}$$

$$\begin{aligned}K_{2,0}(t_2,t_1,\mathbf{k})_0&=\tfrac{1}{2}(x\Omega)\int_0^\beta ds\,{}^{t_2t_1}\!\begin{bmatrix}\mathbf{k}&-\mathbf{k}\\0&0\end{bmatrix}_{s}[G_{\text{in}}(s)]^2\\&\cong\tfrac{1}{2}(x\Omega)\int_0^\beta ds\,{}^{t_2t_1}\!\begin{bmatrix}\mathbf{k}&-\mathbf{k}\\0&0\end{bmatrix}_{s},\end{aligned} \tag{17c}$$

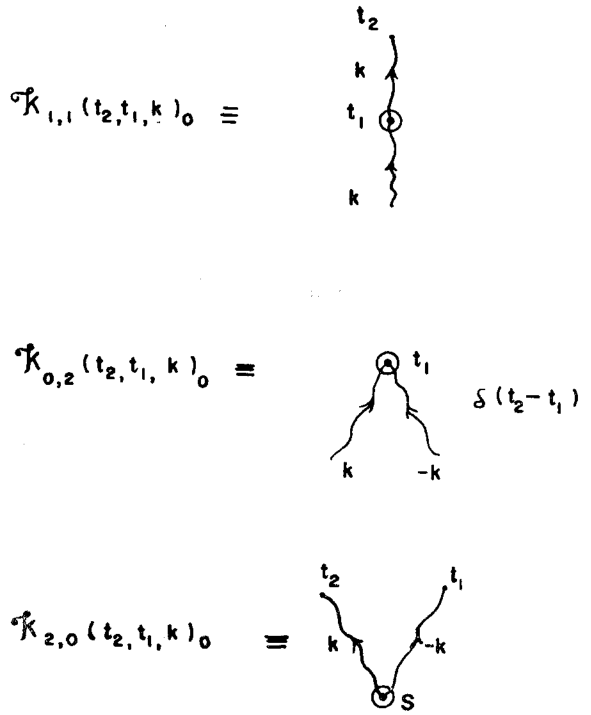

FIG. 1. The lowest order approximations to the functions $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$, where $\mu+\nu=2$, are the three one-vertex master (μ,ν) L graphs shown.

where we have approximated the zero-momentum factors (Sec. 5 in MI) by $G_{\text{out}}(s)\cong\delta(\beta-s)$ and $G_{\text{in}}(s)\cong 1$. By using the explicit form, (1) and (6) for the pair function, and setting $G(t_2,t_1,\mathbf{k})\cong\delta(t_2-t_1)$ in Eq. (9), one can demonstrate that Eqs. (17) lead to terms of the form (16). The question as to whether or not the terms (17) constitute a *good* first approximation to the $K_{\mu,\nu}(t_2,t_1,\mathbf{k})$ will be dealt with at the end of Sec. 5 and then again in the third paper of this series. The approximation $G(t_2,t_1,\mathbf{k})\cong\delta(t_2-t_1)$ and the zero-momentum factor approximations will also be considered more carefully as our analysis progresses.

Consider now the consequence of substituting Eq. (16) for $P_0(t_2,t_1,\mathbf{k})$ into the integral Eq. (12). We shall define the corresponding solution to this integral equation to be $L_0(t_2,t_1,\mathbf{k})$. Then

$$L_0(t_2,t_1,\mathbf{k})\equiv\int_0^\beta ds G_0(t_2,s,\mathbf{k})P_0(s,t_1,\mathbf{k}),\qquad G_0(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+L_0(t_2,t_1,\mathbf{k}).\tag{18}$$

It will be shown in the following section that the solution to the integral equation (18) involves temperature exponentials, and that *all* of the terms of (16) contribute in some way or other to these temperature exponentials. Let us suppose that a particular temperature exponential in the solution to (18) is $\exp(t_2\Delta)$, where $\Delta(\mathbf{k})$ is well defined in the limit $\beta\to\infty$. Then, at very low temperatures, regardless of the sign of Δ, no power series expansion of this temperature exponential is valid. Therefore, the existence of terms of the form (16) immediately implies that the iterative solution to Eq. (12) is not valid at very low temperatures. This analysis, therefore, has already demonstrated the *necessity* of the careful study of the self-energy structure of the quantum statistical theory in MI which led to the integral equations (12)–(15) and the master-graph formulation.

We shall return to the analysis of this self-energy structure in Sec. 5 after studying the integral equation (18) and some of its consequences in the next two sections. The role in the theory of the part of $P(t_2,t_1,\mathbf{k})$, which is not of the form $P_0(t_2,t_1,\mathbf{k})$, will also be clarified in Sec. 5 [see Eq. (60) and below].

3. FUNDAMENTAL INTEGRAL EQUATION

The fundamental integral equation in the analysis of this paper is Eq. (18), with the kernel (16). At the end of the preceding section, we have demonstrated that at very low temperatures no iterative expansion of this integral equation is valid. Therefore, this equation must be solved, and in fact, we shall now give the exact solution. We shall not go through the derivation of this solution, but rather, we shall write the solution down and then indicate how to verify it *a posteriori*. Thus, one finds that

$$\begin{aligned}G_0(t_2,t_1)=(1+B)\{&[\delta(t_2-t_1)+\Delta_+C_+^{(>)}(t_2)e^{-t_1\Delta_+}\\&-\Delta_-C_-^{(>)}(t_2)e^{-t_1\Delta_-}]\theta(t_2-t_1)\\&+[\Delta_+C_+^{(<)}(t_2)e^{-t_1\Delta_+}-\Delta_-C_-^{(<)}(t_2)e^{-t_1\Delta_-}\\&+B^{(<)}(t_2)\delta(\beta-t_1)]\theta(t_1-t_2)\},\end{aligned}\tag{19}$$

where the quantity B appears *only* in the over-all multiplying factor $(1+B)$ when this factor is also introduced on the left-hand side of (16). In Eq. (19) and throughout the rest of the paper, we shall frequently omit the dependence on $\mathbf{k}$ from the notation for the various quantities. Of course, this can only be done when there is only one momentum variable in an equation.

The proof that the solution (19) is correct can be made by substituting Eq. (19) into Eq. (18) and then performing all of the integrations on the right-hand side of (18). One then matches the coefficients of similar t_1-dependent exponentials on both sides of (18) for the two cases $t_1>t_2$ and $t_1<t_2$. This gives the following five identities:

$$[C_+^{(>)}(t_2)-C_+^{(<)}(t_2)]e^{-t_2\Delta_+}=1+[C_-^{(>)}(t_2)-C_-^{(<)}(t_2)]e^{-t_2\Delta_-},\tag{20}$$

$$C_+^{(<)}(t_2)e^{-\beta\Delta_+}=C_-^{(<)}(t_2)e^{-\beta\Delta_-}+B^{(<)}(t_2),\tag{21}$$

$$[\Delta_+(\Delta_+-D)^{-1}C-C']C_+^{(>)}(t_2)=[\Delta_-(\Delta_--D)^{-1}C-C']C_-^{(>)}(t_2),\tag{22}$$

$$\Delta_+(\Delta_+-D)^{-1}[C_+^{(>)}(t_2)-C_+^{(<)}(t_2)]e^{-t_2\Delta_+}=1+\Delta_-(\Delta_--D)^{-1}[C_-^{(>)}(t_2)-C_-^{(<)}(t_2)]e^{-t_2\Delta_-},\tag{23}$$

$$[B'+\Delta_+(\Delta_+-D)^{-1}B'']C_+^{(>)}(t_2) -[B'+\Delta_-(\Delta_--D)^{-1}B'']C_-^{(>)}(t_2) = e^{\beta D}\{[1+D(\Delta_+-D)^{-1}B'']C_+^{(<)}(t_2)e^{-\beta\Delta_+} -[1+D(\Delta_--D)^{-1}B'']C_-^{(<)}(t_2)e^{-\beta\Delta_-}\}. \quad (24)$$

One also obtains a single equation for the determination of Δ_+ and Δ_-,

$$\Delta_\pm{}^2-(A+D)\Delta_\pm+(A+C)D=0\,, \quad (25)$$

which has the solutions

$$\Delta_\pm=\tfrac{1}{2}(A+D)\mp\tfrac{1}{2}[(A-D)^2-4CD]^{1/2}. \quad (26)$$

It should now be clear that the solution (19) for $G_0(t_2,t_1)$ is correct because the five quantities $C_\pm^{(>)}$, $C_\pm^{(<)}$, and $B^{(<)}$ are completely determined by the five linearly independent equations (20)–(24). These simple algebraic equations can be easily solved, and we shall write down the solution in Sec. 9 for the special case when the B's are all zero. For the rest of this paper, however, it is not necessary to have explicit expressions for the quantities in the solution (19). It is only sufficient to know the general form (19) of the solution for $G_0(t_2,t_1)$.

There is another reason why it is inappropriate to discuss the solution to Eqs. (20)–(24) here. This reason is connected with the fact that whereas the Λ transformation of Sec. 5 leaves the form of $G_0(t_2,t_1)$ invariant; at the same time it causes one to focus attention on a new quantity $\Lambda(t_2,t_1)$ and its relation to $G_0(t_2,t_1)$ rather than on G_0, P_0, and the integral Eq. (18). This very subtle point will be discussed in detail at the end of Sec. 4 and below Eq. (53), but for the present it is sufficient to point out that this subtle point has the effect of changing the identities (22)–(24). Only the *basic* identities (20) and (21) remain invariant under the Λ transformation, and the reason for this is because they do not depend on the particular quantities in $P_0(t_2,t_1)$, Eq. (16), except through the functions Δ_+ and Δ_-. It will be seen in Sec. 4 that the identities (20) and (21) are essential to our analysis.

We next write down an important integral of $G_0(t_2,t_1)$.

$$\int_{t_0}^{\beta} dt_1 G_0(t_2,t_1) =(1+B)\{[C_+^{(>)}(t_2)e^{-t_0\Delta_+}-C_-^{(>)}(t_2)e^{-t_0\Delta_-}]\theta(t_2-t_0) +[C_+^{(<)}(t_2)e^{-t_0\Delta_+}-C_-^{(<)}(t_2)e^{-t_0\Delta_-}]\theta(t_0-t_2)\}. \quad (27)$$

In the evaluation of this integral we have used the identities (20) and (21). For the special case when $t_0=0$, we define this integral times a factor to be a quantity $\zeta(t_2,\mathbf{k})$. Then

$$\zeta(t_2,\mathbf{k})\equiv[1+B(\mathbf{k})]^{-1}\exp[-t_2(\omega_{\mathbf{k}}+\Delta^{(0)})]\times\int_0^\beta dt_1 G_0(t_2,t_1,\mathbf{k}) =\exp[-t_2(\omega_{\mathbf{k}}+\Delta^{(0)})]\times[C_+^{(>)}(t_2,\mathbf{k})-C_-^{(>)}(t_2,\mathbf{k})], \quad (28)$$

where the function $\Delta^{(0)}$ will be defined in Sec. 7 [see also Eqs. (33) and (34)].

We conclude this section by discussing two special cases of the above results. The first of these cases occurs when one considers the integral equation (11) instead of (12). In this case we define the special class of terms in $K_{1,1}(t_2,t_1,\mathbf{k})$ [which result in a temperature exponential solution to the integral equation (11)] to be $K_0^{(1)}(t_2,t_1,\mathbf{k})$. The general form of $K_0^{(1)}(t_2,t_1,\mathbf{k})$ is

$$K_0^{(1)}(t_2,t_1,\mathbf{k})=[1+B^{(1)}(\mathbf{k})]^{-1}\times[\Delta^{(1)}(\mathbf{k})+B^{(1)}(\mathbf{k})\delta(t_2-t_1)]\theta(t_2-t_1)\,. \quad (29)$$

One can easily verify that terms of the form (29) occur in $K_{1,1}(t_2,t_1,\mathbf{k})$ by substituting Eq. (6) into Eq. (17a).

We shall define the solution to the integral equation (11) which results when $K_0^{(1)}(t_2,t_1,\mathbf{k})$ is used for the kernel to be $L_0^{(1)}(t_2,t_1,\mathbf{k})$. Then

$$L_0^{(1)}(t_2,t_1,\mathbf{k})\equiv\int_0^\beta ds G_0^{(1)}(t_2,s,\mathbf{k})K_0^{(1)}(s,t_1,\mathbf{k})\,,$$
$$G_0^{(1)}(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+L_0^{(1)}(t_2,t_1,\mathbf{k})\,. \quad (30)$$

The solution to Eq. (30) is readily found to be

$$G_0^{(1)}(t_2,t_1)=[1+B^{(1)}]\times[\delta(t_2-t_1)+\Delta^{(1)}\exp(t_2-t_1)\Delta^{(1)}]\theta(t_2-t_1)\,, \quad (31)$$

and in this case the integral corresponding to Eq. (27) is

$$\int_{t_0}^\beta dt_1 G_0^{(1)}(t_2,t_1) =[1+B^{(1)}]\exp[(t_2-t_0)\Delta^{(1)}]\theta(t_2-t_0)\,. \quad (32)$$

The other special case occurs only when $\mathbf{k}=0$, and this case will be discussed in detail in Sec. 7 when the zero-momentum factors of master graphs are considered. In this case, there is no integral equation such as (18) or (30) to motivate our determination of $G_0^{(0)}(t_2,t_1)$. Rather, we shall merely define this quantity to be of the form (31):

$$G_0^{(0)}(t_2,t_1)\equiv[1+B^{(0)}]\times[\delta(t_2-t_1)+\Delta^{(0)}\exp(t_2-t_1)\Delta^{(0)}]\theta(t_2-t_1)\,, \quad (33)$$

with

$$\int_{t_0}^\beta dt_1 G_0^{(0)}(t_2,t_1) =[1+B^{(0)}]\exp[(t_2-t_0)\Delta^{(0)}]\theta(t_2-t_0)\,. \quad (34)$$

The definitions of the quantities $\Delta^{(0)}$ and $B^{(0)}$ will be made in Sec. 7.

We finally observe that the functions $G_0^{(1)}(t_2,t_1)$ and $G_0^{(0)}(t_2,t_1)$ of Eqs. (31) and (33) can be considered to be special cases of the function $G_0(t_2,t_1)$, Eq. (19). One has merely to set $C_\pm^{(<)}=B^{(<)}=C_-^{(>)}=0$ in Eq. (19) and to set $C_+^{(>)}(t_2)=e^{t_2\Delta}$. If the appropriate superscript (1) or (0) is then attached to the quantities Δ and B, then one obtains either $G_0^{(1)}(t_2,t_1)$ or $G_0^{(0)}(t_2,t_1)$.

4. PAIR-FUNCTION TRANSFORMATION

In Sec. 3 we have arrived at explicit expressions for $G_0(t_2,t_1)$ and $G_0^{(1)}(t_2,t_1)$. The question now arises as to how these two (and also two other) functions are to be used in the theory. The obvious answer to this question is to use them as first approximations to the functions $G_{1,1}(t_2,t_1)$ and $G(t_2,t_1)$, and to then start calculating thermodynamic quantities and distribution functions. But in the master graphs, the functions $G_{1,1}$ and G usually appear multiplied by pair functions, and these products are to be integrated over temperature variables. Thus, we are led to consider such integrals, and the consequences of this consideration then motivates the Λ transformation of the next section.

In anticipation of the next section, we now define a *transformed pair function* as follows:

$$ {}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} \equiv \zeta(t_0,\mathbf{k}_3)\zeta(t_0,\mathbf{k}_4)\zeta^{-1}(t_1,\mathbf{k}_1)\zeta^{-1}(t_2,\mathbf{k}_2)\exp[t_0(\omega_3+\omega_4+2\Delta^{(0)})] \times\exp[-t_1(\omega_1+\Delta^{(0)})]\exp[-t_2(\omega_2+\Delta^{(0)})]\int_0^\beta ds_1ds_2G_0(t_1,s_1,\mathbf{k}_1)G_0(t_2,s_2,\mathbf{k}_2)\,{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}, \quad (35) $$

where the function $\zeta(t,\mathbf{k})$ is defined by Eq. (28) and $\omega_i=\omega(\mathbf{k}_i)$. Aside from the over-all multiplying factors, the definition (35) consists of precisely the kind of integrations referred to in the preceding paragraph. By using Eq. (1), this expression can be rewritten in the following form:

$$ {}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} = \zeta(t_0,\mathbf{k}_3)\zeta(t_0,\mathbf{k}_4)\zeta^{-1}(t_1,\mathbf{k}_1)\zeta^{-1}(t_2,\mathbf{k}_2)\exp[t_0(\omega_3+\omega_4+2\Delta^{(0)})]\exp[-t_1(\omega_1+\Delta^{(0)})]\exp[-t_2(\omega_2+\Delta^{(0)})] \times\left\{\int_{t_0}^\beta ds_1G_0(t_1,s_1,\mathbf{k}_1)\,{}^{s_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\int_{s_1}^\beta ds_2G_0(t_2,s_2,\mathbf{k}_2)+\int_{t_0}^{\beta(-)} ds_2G_0(t_2,s_2,\mathbf{k}_2)\,{}^{s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\int_{s_2}^\beta ds_1G_0(t_1,s_1,\mathbf{k}_1)\right\}. \quad (36) $$

We note that if the s_1 integrations are performed last in *both* of the two terms of Eq. (36); then one will always have $s_2<s_1<\beta$ in the second term. Therefore, the $B^{(<)}(t_2)$ part of the second term will make no contribution. To account for this situation we have written the upper s_2 integration limit of the second term as $\beta^{(-)}$. Of course, we could just as well attach the superscript $(-)$ to the upper s_1 integration limit of the first term, there being no difference in the final result.

We wish to perform the integrations in Eq. (36). Two of them can be accomplished immediately by substituting Eq. (27). One must then substitute Eq. (19) and perform the remaining single integrations, of which there are many. Fortunately, except for simple δ-function integrations, these remaining integrations can all be accomplished with the aid of a single identity; namely,

$$ [\Delta_\pm(\mathbf{k}_1)+\Delta_\pm(\mathbf{k}_2)]\int_{t_0}^{t_1} ds_1\exp[-s_1(\Delta_\pm(\mathbf{k}_1)+\Delta_\pm(\mathbf{k}_2))]\,{}^{s_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} = {}_{(\pm,\pm)}^{\;t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\exp[-t_0(\omega_3+\omega_4+2\Delta^{(0)})]-\exp[-t_1(\Delta_\pm(\mathbf{k}_1)+\Delta_\pm(\mathbf{k}_2))]\,{}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}, \quad (37) $$

where

$$ {}_{(i,j)}^{\;t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} \equiv \exp[t_0(\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2))]g_{i,j}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\sum_{\mathbf{k}_5\mathbf{k}_6}\exp[t_1(\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6))] \times\exp[t_0(\epsilon(\mathbf{k}_5)+\epsilon(\mathbf{k}_6))]f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)P\left(\frac{1}{\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)+\delta(t_1-t_0) \times\exp[t_0(\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2))]f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4), \quad (38) $$

$$ g_{i,j}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\equiv f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+[\Delta_i(\mathbf{k}_1)+\Delta_j(\mathbf{k}_2)]f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4) +\sum_{\mathbf{k}_5\mathbf{k}_6}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)\left[P\left(\frac{1}{\epsilon(\mathbf{k}_1)+\epsilon(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)-P\left(\frac{1}{\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)\right], \quad (39) $$

$$ \begin{aligned} &\epsilon_\pm(\mathbf{k})\equiv\omega(\mathbf{k})+\Delta^{(0)}-\Delta_\pm(\mathbf{k})=\epsilon(\mathbf{k})-\Delta_\pm(\mathbf{k}), \qquad \epsilon(\mathbf{k})\equiv\omega(\mathbf{k})+\Delta^{(0)},\\ &\epsilon_1(\mathbf{k})\equiv\omega(\mathbf{k})+\Delta^{(0)}-\Delta^{(1)}(\mathbf{k})=\epsilon(\mathbf{k})-\Delta^{(1)}(\mathbf{k}), \qquad \epsilon_0\equiv 0. \end{aligned} \quad (40) $$

The proof of Eq. (37) can be made by simply substituting Eq. (6) into the left-hand side. It should be observed that there are four possible applications of this identity corresponding to the two pairs of ($\pm$) signs. The quantities $\epsilon_1(\mathbf{k})$ and ϵ_0 defined in Eqs. (40) have not occurred in any of the preceding expressions, but we have included them because they will be encountered below. We finally remark that the introduction of the ϵ's, Eqs. (40), has not been done to simplify Eq. (38) or (39), but rather to simplify subsequent expressions and their eventual interpretation.

Even with the aid of Eq. (37), it is extremely tedious to derive a final useful expression for the transformed pair function (36). We shall omit the numerous albegraic manipulations and give only the final result.

$$ {}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} = \zeta(t_0,\mathbf{k}_3)\zeta(t_0,\mathbf{k}_4)\zeta^{-1}(t_1,\mathbf{k}_1)\zeta^{-1}(t_2,\mathbf{k}_2)\sum_{i=+,-} i\left\{A_i^{(<)}(t_1,\mathbf{k}_1)\;{}^{t_1t_2}_{(i,<)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} + A_i^{(>)}(t_1,\mathbf{k}_1)\;{}^{t_1t_2}_{(i,>)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\right\}, \tag{41}$$

where, with $i=+$ or $-$,

$$\begin{aligned} A_i^{(<)}(t,\mathbf{k}) &\equiv [1+B(\mathbf{k})]\exp[-t\epsilon(\mathbf{k})]C_i^{(<)}(t,\mathbf{k}),\\ A_i^{(>)}(t,\mathbf{k}) &\equiv [1+B(\mathbf{k})]\exp[-t\epsilon(\mathbf{k})]C_i^{(>)}(t,\mathbf{k}), \end{aligned} \tag{42}$$

$$\begin{aligned} {}^{t_1t_2}_{(i,<)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} \equiv \sum_{j=+,-} j\Bigg[& A_j^{(<)}(t_2,\mathbf{k}_2)\left\{ {}^{\beta}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} - {}^{t_2}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_2-t_1)\theta(t_2-t_0) - {}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_2)\theta(t_1-t_0)\right\}\\ & + A_j^{(>)}(t_2,\mathbf{k}_2)\left\{ {}^{t_2}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_2-t_0) - {}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0)\right\}\theta(t_2-t_1)\Bigg] \quad \text{if } t_1\neq t_2\\ \equiv \sum_{j=+,-} j & A_j^{(<)}(t_1,\mathbf{k}_2)\left\{ {}^{\beta}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} - {}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0)\right\} \quad \text{if } t_1=t_2, \end{aligned} \tag{43}$$

$$\begin{aligned} {}^{t_1t_2}_{(i,>)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0} \equiv \sum_{j=+,-} j\Bigg[& A_j^{(<)}(t_2,\mathbf{k}_2)\left\{ {}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0) - {}^{t_2}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_2-t_0)\right\}\theta(t_1-t_2)\\ & + A_j^{(>)}(t_2,\mathbf{k}_2)\left\{ {}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_2-t_1) + {}^{t_2}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_2)\right\}\theta(t_1-t_0)\theta(t_2-t_0)\Bigg] \quad \text{if } t_1\neq t_2\\ \equiv \sum_{j=+,-} j & A_j^{(>)}(t_1,\mathbf{k}_2)\;{}^{t_1}_{(i,j)}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0) \quad \text{if } t_1=t_2. \end{aligned} \tag{44}$$

In Eq. (41), attention has been focused on the variables t_1 and $\mathbf{k}_1$, thereby apparently destroying the symmetry of the transformed pair function. Thus, from Eq. (35) it is easy to see that the transformed pair function is invariant under the interchange $(t_1,k_1,k_3) \rightleftharpoons (t_2,k_2,k_4)$. However, one can verify that this invariance is still present in Eq. (41) when Eqs. (43) and (44) are substituted. One should also observe that the untransformed pair functions of the second term in (37) have all canceled out in the final expression (41). This cancellation has been achieved with the aid of the identities (20) and (21).

Equation (41) can be greatly simplified in the important case when only the first term of Eq. (38) is retained. One then obtains the approximate expression

$$\begin{aligned} {}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} = {} & \zeta(t_0,\mathbf{k}_3)\zeta(t_0,\mathbf{k}_4)\zeta^{-1}(t_1,\mathbf{k}_1)\zeta^{-1}(t_2,\mathbf{k}_2)\sum_{i=+,-} i[A_i^{(<)}(t_1,\mathbf{k}_1)\theta(t_0-t_1)+A_i^{(>)}(t_1,\mathbf{k}_1)\theta(t_1-t_0)]\\ & \times\left[{}^{t_2}_{(i,<)}\begin{pmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{pmatrix}_{t_0}\theta(t_0-t_2) + {}^{t_2}_{(i,>)}\begin{pmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{pmatrix}_{t_0}\theta(t_2-t_0)\right], \end{aligned} \tag{45}$$

where

$$\begin{aligned} {}^{t_2}_{(i,<)}\begin{pmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{pmatrix}_{t_0} &\equiv \sum_{j=+,-} jA_j^{(<)}(t_2,\mathbf{k}_2)g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\exp[t_0(\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2))],\\ {}^{t_2}_{(i,>)}\begin{pmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{pmatrix}_{t_0} &\equiv \sum_{j=+,-} jA_j^{(>)}(t_2,\mathbf{k}_2)g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\exp[t_0(\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2))]. \end{aligned} \tag{46}$$

The expression (28) for $\zeta(t,\mathbf{k})$ can also be simplified by using the second of the definitions (42). This gives the result

$$\zeta(t,\mathbf{k})=[1+B(\mathbf{k})]^{-1}[A_+^{(>)}(t,\mathbf{k})-A_-^{(>)}(t,\mathbf{k})]. \quad (47)$$

We next discuss the special cases which occur when either or both of the G_0 functions in Eqs. (35) is replaced by a $G_0^{(1)}$ or a $G_0^{(0)}$ function. According to the last paragraph of Sec. 3, one can immediately obtain either of these cases from Eqs. (41)–(47) by setting $A_\pm^{(<)}=A_-^{(>)}=0$. The quantity $A_+^{(>)}(t_2,\mathbf{k})$ then becomes either $A^{(1)}(t_2,\mathbf{k})$ or $A^{(0)}$, where [see also the second of Eqs. (42)]

$$\begin{aligned}A^{(1)}(t_2,\mathbf{k})&\equiv[1+B^{(1)}(\mathbf{k})]\exp[-t_2\epsilon_1(\mathbf{k})],\\ A^{(0)}&\equiv[1+B^{(0)}].\end{aligned} \quad (48)$$

Here, the quantity $\epsilon_1(\mathbf{k})$ is defined by the third of Eqs. (40). Similarly, the function $\zeta(t,\mathbf{k})$ becomes either $\zeta^{(1)}(t,\mathbf{k})$ or $\zeta^{(0)}$, where

$$\begin{aligned}\zeta^{(1)}(t,\mathbf{k})&\equiv\exp[-t_2\epsilon_1(\mathbf{k})],\\ \zeta^{(0)}(t)&\equiv 1.\end{aligned} \quad (49)$$

It should be noted that *either* the outgoing or the incoming $\zeta(t)$ factors in Eq. (41) can be replaced by one of Eqs. (49), the particular choice being determined as indicated below Eq. (65). Finally, the $(\pm)$ signs of Eqs. (38), (39), etc., must be replaced by either of the indices (0) or (1) when $G_0\to G_0^{(0)}$ or $G_0\to G_0^{(1)}$. One must then refer to the last two lines of Eqs. (40) for the appropriate ϵ quantity.

Motivation for the Λ Transformation

We now return to the discussion below Eq. (26) and observe that if one computes the function $P_0(t_2,t_1,\mathbf{k})$ of Eq. (16) *before* the integral (35) is performed, then one will get some result. However, the integral (35) has the effect of introducing more terms of the "type" $P_0(t_2,t_1,\mathbf{k})$ into the theory. This last assertion can be qualitatively verified by comparing Eqs. (6) and (38) and observing that a major difference is the replacement of f_1 in (6) by $g_{i,j}$ in (38). Of course, the temperature exponentials have also been changed in (38), and this suggests that the very basis for emphasizing the importance of the integral Eq. (18) has been destroyed by the integral (35).

The confusion of the preceding paragraph is intentional, for the important point is that as soon as one performs the integral (35) to obtain Eqs. (41)–(44) and (38)–(40), one has eliminated the role of the integral Eq. (18). The important terms in the theory are no longer those of the form $P_0(t_2,t_1,\mathbf{k})$, Eq. (16), but rather, a new set of terms of the form $\Lambda(t_2,t_1,\mathbf{k})$. The question now is: How can one identify those terms $\Lambda(t_2,t_1,\mathbf{k})$ in the theory, which, *after* the integral (35) is performed, give the dominant contribution to the theory at very low temperatures? This difficult problem is solved by performing the Λ *transformation* on the entire theory. After the Λ transformation, it is then easy to choose the function $\Lambda(t_2,t_1,\mathbf{k})$ in such a way that iterative solutions to integral equations of the type (11) and (12) become valid. It remains to be said that the difficulty of properly motivating the Λ transformation is mostly due to the fact that one really only understands it by hindsight.

5. Λ TRANSFORMATION

Section 4 has been devoted to the study of the integral (35), and the results of that section are very important for any application of the theory to a real or model Bose system. None of these detailed results are required in the formal equations of the present section, however, for only the insight which they have provided is necessary here. Thus, we shall begin by defining the function $\Lambda(t_2,t_1,\mathbf{k})$ as follows:

$$\Lambda(t_2,t_1,\mathbf{k})\equiv e^{-t_2\epsilon(\mathbf{k})}\zeta^{-1}(t_2,\mathbf{k})\times[G_0(t_2,t_1,\mathbf{k})-\delta(t_2-t_1)]\zeta(t_1,\mathbf{k})e^{t_1\epsilon(\mathbf{k})}. \quad (50)$$

With the aid of Eqs. (19), (40), and (42), this definition can be rewritten as

$$\begin{aligned}\Lambda(t_2,t_1,\mathbf{k})=\zeta^{-1}(t_2)\zeta(t_1)\{&[B\delta(t_2-t_1)+\Delta_+A_+^{(>)}(t_2)e^{t_1\epsilon+}\\ &-\Delta_-A_-^{(>)}(t_2)e^{t_1\epsilon-}]\theta(t_2-t_1)\\ &+[\Delta_+A_+^{(<)}(t_2)e^{t_1\epsilon+}-\Delta_-A_-^{(<)}(t_2)e^{t_1\epsilon-}\\ &+B_0(t_2)\delta(\beta-t_1)]\theta(t_1-t_2)\}, \quad (51)\end{aligned}$$

where

$$B_0(t_2)\equiv(1+B)B^{(<)}(t_2)e^{(\beta-t_2)\epsilon}. \quad (52)$$

We shall see that the function $\Lambda(t_2,t_1,\mathbf{k})$ plays the same role *after* the Λ transformation as the function $P_0(t_2,t_1,\mathbf{k})$, Eq. (16), plays *before* the Λ transformation. Moreover, Eq. (50), which gives the relation between G_0 and Λ, replaces Eq. (18), which gives the relation between G_0 and P_0. The relation (50) can be solved for $G_0(t_2,t_1)$ as follows:

$$G_0(t_2,t_1,\mathbf{k})=\delta(t_2-t_1)+e^{t_2\epsilon(\mathbf{k})}\zeta(t_2,\mathbf{k})\times\Lambda(t_2,t_1,\mathbf{k})\zeta^{-1}(t_1,\mathbf{k})e^{-t_1\epsilon(\mathbf{k})}. \quad (53)$$

It is to be emphasized that we are now considering the function $G_0(t_2,t_1,\mathbf{k})$ of Eq. (19) to be a *given* function, to be used in the integral (35) and elsewhere (see below). The various quantities on the right-hand side of (19) or (51) are now *unknown* functions to be determined after the Λ transformation in any application of the theory to a particular system. As has been emphasized in Sec. 3, the identities (20) and (21) must continue to be valid after the Λ transformation, for otherwise the derivations of Sec. 4 breakdown. We therefore *assume* that the identities (20) and (21) are always correct for any system. On the other hand, the identities (22)–(24) and Eq. (25) for the determination of Δ_+ and Δ_- *will not* be valid in general. In the following paper, however, we shall show that one obtains these identities, even after the Λ transformation, in the first approximation to the thermodynamics of a dilute gas of hard-sphere bosons. Of course, the various quantities A, B, C, etc., must be determined in this case by an actual calculation of the lowest order graphs (Fig. 1) of the theory.

In a previous paper[6] we have shown that the momentum space ordering in a very low-temperature Fermi gas is explicitly exhibited in a quantum statistical theory by the application of a Λ transformation. The formalism developed in that paper can also be readily applied to a degenerate Bose system above the critical temperature; i.e., when $\langle x\rangle=0$. The equations for the Λ transformation in the $\langle x\rangle=0$ (Bose) case form the basis for the Λ transformation in the present case when $\langle x\rangle\neq 0$.

The basic equation of the Λ transformation when $\langle x\rangle\neq 0$ is

$$G_{1,1}(t_2,t_1)=\zeta(t_2)e^{t_2\epsilon}\int_0^\beta dsG_{1,1}'(t_2,s)\zeta^{-1}(s)e^{-s\epsilon}G_0(s,t_1)\,, \quad (54)$$

in which a new function $G_{1,1}'(t_2,t_1,\mathbf{k})$ is defined. If one also defines $L_{1,1}'(t_2,t_1,\mathbf{k})$ by the equation

$$G_{1,1}'(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+L_{1,1}'(t_2,t_1,\mathbf{k})\,, \quad (55)$$

then Eq. (54) can be rearranged to the form

$$L_{1,1}'(t_2,t_1)=\zeta^{-1}(t_2)e^{-t_2\epsilon}L_{1,1}(t_2,t_1)\zeta(t_1)e^{t_1\epsilon}-\int_0^\beta dsG_{1,1}'(t_2,s)\Lambda(s,t_1)\,, \quad (56)$$

where Eqs. (15) and (53) have been used. One next substitutes Eq. (54) into Eq. (12) to obtain

$$L_{1,1}(t_2,t_1)=\zeta(t_2)e^{t_2\epsilon}\times\int_0^\beta dsG_{1,1}'(t_2,s)\mathcal{P}'(s,t_1)\zeta^{-1}(t_1)e^{-t_1\epsilon}, \quad (57)$$

where

$$\mathcal{P}'(t_2,t_1)\equiv\zeta^{-1}(t_2)e^{-t_2\epsilon}\int_0^\beta dsG_0(t_2,s)P(s,t_1)\zeta(t_1)e^{t_1\epsilon}. \quad (58)$$

Finally, one substitutes Eq. (57) into Eq. (56), thereby arriving at an important consequence of the definition (54).

$$L_{1,1}'(t_2,t_1)=\int_0^\beta dsG_{1,1}'(t_2,s)P'(s,t_1)\,, \quad (59)$$

where

$$P'(t_2,t_1)\equiv\mathcal{P}'(t_2,t_1)-\Lambda(t_2,t_1)\,. \quad (60)$$

We now observe that Eq. (59) has precisely the same form as Eq. (12), with unprimed quantities replaced by primed quantities. Equation (60), on the other hand, gives us the possibility of subtracting from $\mathcal{P}'(t_2,t_1)$ *all* of those terms which, when iterated, would give large contributions to $L_{1,1}'(t_2,t_1)$ at very low temperatures. But such terms have precisely the form (51) [after the Λ transformation], and therefore the integral Eq. (59) can be solved by iteration in any application; i.e., the function $P'(t_2,t_1)$ consists only of the "small" terms in $\mathcal{P}'(t_2,t_1)$.

Equations (50)–(60) give the essence of the Λ transformation. The many further steps required are only those which are necessary to demonstrate that the Λ transformation is a completely consistent transformation from a set of unprimed quantities to a set of primed quantities. One must also show *explicitly* how all of the primed quantities are to be calculated in order that the theory can be applied to any particular Bose system. For example, one can see from the above equations, that Eq. (58) gives the prescription for calculating $\mathcal{P}'(t_2,t_1)$, and that the other primed quantities are determined as soon as $\mathcal{P}'(t_2,t_1)$ is determined. In fact, upon comparing Eqs. (8) and (9) with Eqs. (35) and (58), one may immediately conclude that the prescription (58) is equivalent to the calculation of $\mathcal{P}'(t_2,t_1)$ by everywhere replacing the pair functions (1) in the theory by the transformed pair function (35). We shall clarify this statement in detail, after first introducing transformed master (μ,ν) graphs.

Transformed Master (μ,ν) Graphs

A *transformed master* (μ,ν) *graph* or a *transformed master* (μ,ν) *L graph* is calculated by using the same rules and diagram as in the calculation of the corresponding untransformed master (μ,ν) graph or master (μ,ν) L graph [Sec. 6 in MI], except for the following changes:

(a) Pair functions (1) are replaced by transformed pair functions (35), *except* for the subtracted wiggly-line double-bond terms of rule (viii) in Sec. 6 of MI.

(b) The line factors $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$ are replaced by the line factors $\mathcal{G}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$.

(c) The outgoing zero-momentum missing line factors $G_{\text{out}}(t)$ are replaced by the factors $\exp[\beta\Delta^{(0)}]\cdot G_{\text{out}}'(t)$; and the incoming zero-momentum missing line factors $G_{\text{in}}(t)$ are replaced by the factors $[1+B^{(0)}]G_{\text{in}}'(t)$.

In order to verify that the second of the above changes is consistent with Eq. (35), one must show that the following relations exist between the $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$ and the $\mathcal{G}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$:

$$\mathcal{G}_{1,1}(t_2,t_1,\mathbf{k})=\zeta(t_2,\mathbf{k})e^{t_2\epsilon(\mathbf{k})}\int_0^\beta ds\mathcal{G}_{1,1}'(t_2,s,\mathbf{k})\times\zeta^{-1}(s,\mathbf{k})e^{-s\epsilon(\mathbf{k})}G_0(s,t_1,\mathbf{k})\,, \quad (61)$$

$$\mathcal{G}_{0,2}(t_2,t_1,\mathbf{k})=\int_0^\beta ds_1ds_2\mathcal{G}_{0,2}'(s_2,s_1,\mathbf{k})\times\zeta^{-1}(s_2,\mathbf{k})e^{-s_2\epsilon(\mathbf{k})}e^{-s_1\Delta^{(1)}(-\mathbf{k})}\times G_0(s_2,t_2,\mathbf{k})G_0^{(1)}(s_1,\,t_1,\,-\mathbf{k})\,, \quad (62)$$

$$\mathcal{G}_{2,0}(t_2,t_1,\mathbf{k})=\zeta(t_2,\mathbf{k})e^{t_2\epsilon(\mathbf{k})}e^{t_1\Delta^{(1)}(-\mathbf{k})}\mathcal{G}_{2,0}'(t_2,t_1,\mathbf{k})\,. \quad (63)$$

Equations (61)–(63) assure that the pair functions are all transformed by the correct $G_0(t_2,t_1)$ functions. These equations will be investigated in detail in the following section and the appearance of the $G_0^{(1)}(t_2,t_1)$ function in Eq. (62) will be clarified in this section.

In order to verify that the change (c) for the transformed master graphs is consistent with Eq. (35), one must demonstrate the validity of the following relations:

$$G_{\mathrm{out}}(t)=\exp[\beta\Delta^{(0)}]\int_0^\beta dsG_{\mathrm{out}}'(s)\times\exp[-s\Delta^{(0)}]G_0^{(0)}(s,t), \quad (64)$$

$$G_{\mathrm{in}}(t)=[1+B^{(0)}]\exp[t\Delta^{(0)}]G_{\mathrm{in}}'(t). \quad (65)$$

These two relations will be investigated in detail in Sec. 7.

In Eqs. (62)–(65) we have seen the appearance of the transformation functions $G_0^{(1)}$ and $G_0^{(0)}$ and their associated factors. The *way* in which these functions are to be used in Eq. (35) and the subsequent equations of Sec. 4 is discussed in connection with Eqs. (48) and (49). The decision as to *when* they are to be used will now be discussed. It should be clear that the function $G_0^{(0)}(t_2,t_1)$ of Eq. (33) is only to be used when there is a missing outgoing zero-momentum line at a cluster vertex. When there is an incoming missing zero-momentum line at a vertex, then only the corresponding "incoming" $\zeta(t)$ factor of Eq. (35) is changed. Thus, the $G_0^{(0)}(t_2,t_1)$ function is associated with the transformation of the zero-momentum factors; and the study of this transformation is made in Sec. 7.

The decision as to when a $G_0^{(1)}(t_2,t_1)$ function is to be used, or when an "incoming" $\zeta(t)$ factor is to be replaced by $\zeta^{(1)}(t)$, is determined entirely by whether or not the corresponding line is associated with the function $G(t_2,t_1)$ of Eq. (11). Now, it will be noticed from Eqs. (9), (13), and (14) that the function $G(t_2,t_1)$ can be associated with the $(-\mathbf{k})$ lines in master graphs. Similarly, the function $G_{1,1}(t_2,t_1)$ can be associated with $(+\mathbf{k})$ lines in master graphs. Of course, this association is only true provided that the alternate forms of Eqs. (13) and (14) given in MI are not used. In order to facilitate our analysis, then, we adopt the convention that the function $G(t_2,t_1)$ will always be associated with $-\mathbf{k}$ lines and that the function $G_{1,1}(t_2,t_1)$ will always be associated with $+\mathbf{k}$ lines. (Having arrived at a correct expression, one then need not worry about what happens when the sum over all $\mathbf{k}$ is performed.)

The convention of the preceding paragraph determines when a function $G_0^{(1)}(t_2,t_1)$ is to be used in the transformed pair function (35). It also explains the appearance of the $G_0^{(1)}(t_2,t_1)$ function and its associated exponential factors in Eqs. (62) and (63).

We must now discuss the Λ transformation of the function $G(t_2,t_1,-\mathbf{k})$. The basic equation is, again, one similar to (54).

$$G(t_2,t_1,-\mathbf{k})=\exp[t_2\Delta^{(1)}(-\mathbf{k})]\int_0^\beta dsG'(t_2,s,-\mathbf{k})\times\exp[-s\Delta^{(1)}(-\mathbf{k})]G_0^{(1)}(s,t_1,-\mathbf{k}), \quad (66)$$

in which a new function $G'(t_2,t_1,-\mathbf{k})$ is defined. We also define a function $L'(t_2,t_1,-\mathbf{k})$ by the equation

$$G'(t_2,t_1,-\mathbf{k})\equiv\delta(t_2-t_1)+L'(t_2,t_1,-\mathbf{k}). \quad (67)$$

Equations analogous to Eqs. (56) and (57) can next be written down, but we shall omit them here. Rather, we shall include the important equations, analogous to Eqs. (58)–(60). Thus, if we define a function

$$\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})\equiv\exp[-t_2\Delta^{(1)}(-\mathbf{k})]\int_0^\beta dsG_0^{(1)}(t_2,s,-\mathbf{k})\times K_{1,1}(s,t_1,-\mathbf{k})\exp[t_1\Delta^{(1)}(-\mathbf{k})], \quad (68)$$

then the function $L'(t_2,t_1,-\mathbf{k})$ is given by

$$L'(t_2,t_1,-\mathbf{k})=\int_0^\beta dsG'(t_2,s,-\mathbf{k})K_{1,1}^{(1)\prime}(s,t_1,-\mathbf{k}), \quad (69)$$

where

$$K_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})\equiv\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})-\Lambda^{(1)}(t_2,t_1,-\mathbf{k}). \quad (70)$$

Finally, in analogy with Eqs. (50) and (51), the function $\Lambda^{(1)}(t_2,t_1,-\mathbf{k})$ is given by

$$\Lambda^{(1)}(t_2,t_1,-\mathbf{k})=\exp[-(t_2-t_1)\Delta^{(1)}(-\mathbf{k})]\times[G_0^{(1)}(t_2,t_1,-\mathbf{k})-\delta(t_2-t_1)], \quad (71)$$

$$\Lambda^{(1)}(t_2,t_1,-\mathbf{k})=\{B^{(1)}(-\mathbf{k})\delta(t_2-t_1)+[1+B^{(1)}(-\mathbf{k})]\Delta^{(1)}(-\mathbf{k})\}\theta(t_2-t_1), \quad (72)$$

where we have used Eq. (31) to obtain (72) from (71).

We can give an interpretation of the Λ transformation for Eqs. (66)–(72) similar to that which we have given for Eqs. (50)–(60). We first observe that Eq. (69) has precisely the same form as Eq. (11). Moreover, Eq. (70) permits the subtraction of all of the large terms in $\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})$, of the form $\Lambda^{(1)}(t_2,t_1,-\mathbf{k})$, Eq. (72). Therefore, with this choice of $\Lambda^{(1)}(t_2,t_1,-\mathbf{k})$, the integral Eq. (69) can be solved by iteration in any application.

We return now to the discussion above Eq. (61), and define functions $\mathcal{K}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$ in analogy with Eq. (8).

$$\mathcal{K}_{\mu,\nu}'(t_2,t_1,\mathbf{k})\equiv\sum[\text{all different transformed master }(\mu,\nu)\ L\text{ graphs}]_{\mathbf{k}}, \quad (73)$$

where $\mu+\nu=2$. With this definition, the function $\mathcal{P}'(t_2,t_1,\mathbf{k})$ of Eq. (58) is given in analogy with Eq. (9) by the expression

$$\mathcal{P}'(t_2,t_1,\mathbf{k})=\mathcal{K}_{1,1}'(t_2,t_1,\mathbf{k})+\int_0^\beta ds_1ds_2\,\mathcal{K}_{2,0}'(t_2,s_1,\mathbf{k})\times G'(s_2,s_1,-\mathbf{k})\,\mathcal{K}_{0,2}'(t_1,s_2,\mathbf{k}), \quad (74)$$

where we can also write the functions $\mathcal{K}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$ of

(73) as transformation equations as follows:

$$\mathcal{K}_{1,1}'(t_2,t_1,\mathbf{k})=\zeta^{-1}(t_2,\mathbf{k})\exp[-t_2\epsilon(\mathbf{k})]\int_0^\beta dsG_0(t_2,s,\mathbf{k})\times K_{1,1}(s,t_1,\mathbf{k})\zeta(t_1,\mathbf{k})\exp[t_1\epsilon(\mathbf{k})],\quad(75)$$

$$\mathcal{K}_{0,2}'(t_2,t_1,\mathbf{k})=K_{0,2}(t_2,t_1,\mathbf{k})\zeta(t_2,\mathbf{k})\times\exp[t_2\epsilon(\mathbf{k})+t_1\Delta^{(1)}(-\mathbf{k})],\quad(76)$$

$$\mathcal{K}_{2,0}'(t_2,t_1,\mathbf{k})=\zeta^{-1}(t_2,\mathbf{k})\exp[-t_2\epsilon(\mathbf{k})-t_1\Delta^{(1)}(-\mathbf{k})]\times\int_0^\beta ds_1ds_2G_0(t_2,s_2,\mathbf{k})G_0^{(1)}(t_1,s_1,-\mathbf{k})\times K_{2,0}(s_2,s_1,\mathbf{k}).\quad(77)$$

It can easily be seen that Eqs. (75)–(77) are consistent with Eqs. (35), (73), and (74) and with the convention adopted above Eq. (66). That these three equations are also consistent with Eqs. (61)–(63) will be shown in the next section.

Equation (73) defines the function $\mathcal{K}_{1,1}'(t_2,t_1,\mathbf{k})$, but not the function $\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1\ -\mathbf{k})$, because the transforming G_0 functions in Eqs. (68) and (75) are different. Actually, with the convention adopted above Eq. (66), Eq. (73) gives $\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})$ correctly, but in order to avoid any possibility of confusion we shall give the definiton of $\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})$ separately, as follows:

$\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})$
$\equiv\sum$ [all different transformed master (1,1) L graphs, with the external lines transformed by the functions $G_0^{(1)}(t_2,s,-\mathbf{k})$ and $\zeta^{(1)}(t_1,-\mathbf{k})]_{-\mathbf{k}}$. (78)

With Eqs. (73), (74), and (78), we have specified precisely all of the primed functions which have been introduced by the Λ transformation; Eqs. (54)–(60) and (66)–(70). It only remains to clarify the transformation of the line factors and zero-momentum factors in the next two sections.

At this point we return to Eqs. (60) and (70) and observe that these equations have "eliminated" the terms Λ and $\Lambda^{(1)}$ from the functions $\mathcal{P}'$ and $\mathcal{K}_{1,1}^{(1)\prime}$, respectively. What then has happened to these quantities, i.e., where have they gone? The answer is partly that they have "reappeared" in the transformed pair functions (35). The rest of the answer will be given in Secs. 6 and 8, where we shall find that the functions $\Lambda(t_2,t_1,\mathbf{k})$ and the closely related $\zeta(t_2,\mathbf{k})$ appear explicitly in the expressions for the grand potential and the momentum distribution. In fact, the nature of the Λ transformation is such that no terms are eliminated or lost from the over-all theory. Rather, a rearrangement of terms has occurred, after which an iterative solution to the basic integral equations is possible.

6. LINE FACTOR TRANSFORMATIONS AND THE MOMENTUM DISTRIBUTION

In this section we shall study in detail Eqs. (61)–(63) for the transformation of the line factors $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$ in order to demonstrate their consistency with the other equations of the Λ transformation in Sec. 5. As a part of this study, we shall be concerned with the transformation of the function $N_{\mu,\nu}(\mathbf{p})$, and this, in turn, will lead us to an expression for the momentum distribution in terms of transformed, or primed, quantities.

We begin by writing down the transformation equations of the functions $L_{0,2}(t_2,t_1,\mathbf{k})$ and $L_{2,0}(t_2,t_1,\mathbf{k})$, Eqs. (13) and (14). According to Eqs. (54), (66), (76), and (77), these transformation equations must be of the form

$$L_{0,2}(t_2,t_1,\mathbf{k})=\int_0^\beta ds_2ds_1L_{0,2}'(s_2,s_1,\mathbf{k})\zeta^{-1}(s_2,\mathbf{k})\times\exp[-s_2\epsilon(\mathbf{k})-s_1\Delta^{(1)}(-\mathbf{k})]\times G_0(s_2,t_2,\mathbf{k})G_0^{(1)}(s_1,t_1,-\mathbf{k}),\quad(79)$$

$$L_{2,0}(t_2,t_1,\mathbf{k})=\zeta(t_2,\mathbf{k})\exp[t_2\epsilon(\mathbf{k})+t_1\Delta^{(1)}(-\mathbf{k})]\times L_{2,0}'(t_2,t_1,\mathbf{k}),\quad(80)$$

where

$$L_{0,2}'(t_2,t_1,\mathbf{k})\equiv\int_0^\beta ds_2ds_1\,\mathcal{K}_{0,2}'(s_2,s_1,\mathbf{k})G_{1,1}'(s_2,t_2,\mathbf{k})\times G'(s_1,t_1,-\mathbf{k})-K_{0,2}^{(1)\prime}(t_2,t_1,\mathbf{k}),\quad(81)$$

$$L_{2,0}'(t_2,t_1,\mathbf{k})\equiv\int_0^\beta ds_2ds_1G_{1,1}'(t_2,s_2,\mathbf{k})\times G'(t_1,s_1,-\mathbf{k})\,\mathcal{K}_{2,0}'(s_2,s_1,\mathbf{k})-\delta(t_2,t_1)K_{2,0}^{(1)\prime}(t_2,t_1,\mathbf{k}).\quad(82)$$

The first thing to notice about these equations is that the transformation of the $L_{\mu,\nu}$ is "opposite" to the transformation of the $K_{\mu,\nu}$ by Eqs. (76) and (77) [compare also Eqs. (54) and (58) or (75) for the case $(\mu,\nu)=(1,1)$]. On the other hand, the transformation of the $L_{\mu,\nu}$ and the $\mathcal{G}_{\mu,\nu}$ by Eqs. (62) and (63) is the same, as it must be [compare also Eqs. (54) and (61) for the case $(\mu,\nu)=(1,1)$]. The next thing to notice is that the functions $K_{0,2}^{(1)\prime}$ and $K_{2,0}^{(1)\prime}$ in Eqs. (81) and (82), respectively, must be defined by the expressions

$K_{0,2}^{(1)\prime}(t_2,t_1,\mathbf{k})$
$\equiv$[the part of $\mathcal{K}_{0,2}'(t_2,t_1,\mathbf{k})$ for which both incoming external lines attach at the same vertex, except that the external lines must attach as free particle lines at *both* ends]
$=K_{0,2}^{(1)}(t_2,t_1,\mathbf{k})$, (83)

$$K_{2,0}{}^{(1)\prime}(t_2,t_1,\mathbf{k}) \equiv [\text{the part of } \mathcal{K}_{2,0}'(t_2,t_1,\mathbf{k}) \text{ for which both outgoing external lines attach at the same vertex, except that the external lines must attach as free-particle lines at } both \text{ ends}] = K_{2,0}{}^{(1)}(t_2,t_1,\mathbf{k}). \quad (84)$$

The function $K_{0,2}{}^{(1)\prime}(t_2,t_1,\mathbf{k})$ includes a $\delta(t_2-t_1)$ δ function as a factor, and it must be subtracted in (81) *only* when the incoming external lines also attach at the same vertex at their tail end (see Fig. 6 in MI). Similarly, the function $K_{2,0}{}^{(1)\prime}(t_2,t_1,\mathbf{k})$ must be subtracted in (82) only when the outgoing external lines attach at the same vertex at their head end (see Fig. 7 in MI), but in this case the Kronecker δ, $\delta(t_2,t_1)$, factor in (82) assures this condition. The reason why these external lines attach as free-particle lines so that there are *no* corresponding transformation functions at the two corresponding incoming (or outgoing) positions in Eq. (35) is that there are no associated $G_{1,1}$ or G functions with $K_{0,2}{}^{(1)}$ and $K_{2,0}{}^{(1)}$ [see Eqs. (13) and (14)]. Thus, the external lines cannot be transformed and therefore the equalities of the second lines of (83) and (84) are explained. [The functions $K_{\mu,\nu}{}^{(1)}$ and $K_{\mu,\nu}{}^{(1)\prime}$, where $(\mu,\nu)=(0,2)$ or $(2,0)$, while being equal, are nevertheless calculated differently because of their *internal* structure.] We also observe that since the external lines of $K_{0,2}{}^{(1)\prime}$ and $K_{2,0}{}^{(1)\prime}$ are not transformed, Eqs. (79) and (80) are not quite correct. However, the definitions (83) and (84) insure that no error is made because of the phrase "at *both* ends," which means that the external lines attach at both of their ends as free-particle lines. Thus, one must be careful to use Eqs. (81) and (82) correctly in any application. We finally observe that all of this difficulty with the functions $K_{0,2}{}^{(1)\prime}$ and $K_{2,0}{}^{(1)\prime}$ has its origin in rule (i) for linked-pair (μ,ν) graphs (Sec. 3 in MI), which states that no wiggly-line double bonds may occur in any graph.

We next define three functions $K_{\mu,\nu}'(\mathbf{p})$, for the case $\mathbf{k}=\mathbf{p}\neq 0$,

$$\begin{aligned} K_{1,1}'(\mathbf{p}) &\equiv \int_0^\beta dt_1 L_{1,1}'(\beta,t_1,\mathbf{p}), \\ K_{0,2}'(\mathbf{p}) &\equiv \int_0^\beta dt_2 dt_1 L_{0,2}'(t_2,t_1,\mathbf{p}), \\ K_{2,0}'(\mathbf{p}) &\equiv L_{2,0}'(\beta,\beta,\mathbf{p}). \end{aligned} \quad (85)$$

In this particular case, it must be understood that the functions $K_{0,2}{}^{(1)\prime}$ and $K_{2,0}{}^{(1)\prime}$ *do not* contribute to the $K_{\mu,\nu}'(\mathbf{p})$, because there are no wiggly-line double-bonds involved. Then, according to Eqs. (54), (55), (15), (79), (80), (28), and (32) the relations between the $K_{\mu,\nu}'(\mathbf{p})$ and the corresponding $K_{\mu,\nu}(\mathbf{p})$ of Eq. (54) in MI are

$$\begin{aligned} K_{1,1}(\mathbf{p}) &= [1+B(\mathbf{p})]\zeta(\beta,\mathbf{p}) \\ &\quad \times\exp[\beta\epsilon(\mathbf{p})][1+K_{1,1}'(\mathbf{p})]-1, \\ K_{0,2}(\mathbf{p}) &= [1+B(\mathbf{p})][1+B^{(1)}(-\mathbf{p})]K_{0,2}'(\mathbf{p}), \\ K_{2,0}(\mathbf{p}) &= \zeta(\beta,\mathbf{p})\exp[\beta(\epsilon(\mathbf{p})+\Delta^{(1)}(-\mathbf{p}))]K_{2,0}'(\mathbf{p}). \end{aligned} \quad (86)$$

Equations (86) will be used in conjunction with the functions $N_{\mu,\nu}'(\mathbf{p})$, which we next define in terms of the corresponding unprimed $N_{\mu,\nu}(\mathbf{p})$.

$$\begin{aligned} N_{1,1}'(\mathbf{p}) &\equiv [1+B(\mathbf{p})]\zeta(\beta,\mathbf{p})\exp[\beta\epsilon(\mathbf{p})]N_{1,1}(\mathbf{p}), \\ N_{0,2}'(\mathbf{p}) &\equiv \zeta(\beta,\mathbf{p})\exp[\beta(\epsilon(\mathbf{p})+\Delta^{(1)}(-\mathbf{p}))]N_{0,2}(\mathbf{p}), \\ N_{2,0}'(\mathbf{p}) &\equiv [1+B(\mathbf{p})][1+B^{(1)}(-\mathbf{p})]N_{2,0}(\mathbf{p}). \end{aligned} \quad (87)$$

These definitions have been made in order that simple relations will result between the $N_{\mu,\nu}'(\mathbf{p})$ and the $K_{\mu,\nu}'(\mathbf{p})$.

We now substitute Eqs. (86) and (87) into Eqs. (35)–(37) of MI and obtain the following transformed equations:

$$\begin{aligned} N_{1,1}'(\mathbf{p}) &\equiv \nu'(\mathbf{p})[1+K_{1,1}'(\mathbf{p})N_{1,1}'(\mathbf{p})+K_{0,2}'(\mathbf{p})N_{2,0}'(\mathbf{p})] \\ &= \nu'(\mathbf{p})[1+K_{1,1}'(\mathbf{p})N_{1,1}'(\mathbf{p}) \\ &\qquad +K_{2,0}'(-\mathbf{p})N_{0,2}'(-\mathbf{p})], \end{aligned} \quad (88)$$

$$N_{0,2}'(\mathbf{p}) = \nu'(\mathbf{p})[K_{1,1}'(\mathbf{p})N_{0,2}'(\mathbf{p}) +K_{0,2}'(\mathbf{p})N_{1,1}'(-\mathbf{p})R(\beta,-\mathbf{p})], \quad (89)$$

$$N_{2,0}'(\mathbf{p}) = \nu'(\mathbf{p})[K_{1,1}'(\mathbf{p})N_{2,0}'(\mathbf{p}) +K_{2,0}'(\mathbf{p})N_{1,1}'(-\mathbf{p})R(\beta,-\mathbf{p})], \quad (90)$$

where we have not given the second forms of Eqs. (89) and (90), because they only involve the replacement $\mathbf{p}\rightleftharpoons -\mathbf{p}$. The functions $\nu'(\mathbf{p})$ and $R(t,\mathbf{p})$ introduced into these equations are defined by

$$R(t,\mathbf{p}) \equiv [1+B^{(1)}(\mathbf{p})][1+B(\mathbf{p})]^{-1}\zeta^{-1}(t,\mathbf{p}) \times\exp[-t\epsilon_1(\mathbf{p})], \quad (91)$$

$$\begin{aligned} \nu'(\mathbf{p}) &\equiv [1+B(\mathbf{p})]\zeta(\beta,\mathbf{p})\exp[\beta\epsilon(\mathbf{p})]\nu(\mathbf{p}) \\ &\quad \times\{1+\nu(\mathbf{p})-[1+B(\mathbf{p})]\zeta(\beta,\mathbf{p}) \\ &\qquad \times\exp[\beta\epsilon(\mathbf{p})]\nu(\mathbf{p})\}^{-1} \\ &= [1+B(\mathbf{p})]\zeta(\beta,\mathbf{p})\exp\beta(g+\Delta^{(0)}) \\ &\quad \times\{1-[1+B(\mathbf{p})]\zeta(\beta,\mathbf{p})\exp\beta(g+\Delta^{(0)})\}^{-1}, \end{aligned} \quad (92)$$

where we have used Eqs. (7) and (40) to obtain the second line of (92).

We are now in a position to investigate the consistency of Eqs. (61)–(63) with all of the other equations of the Λ transformation. In fact, by using the explicit expressions for the $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$ [(66)–(72) in MI], one can verify that Eqs. (61)–(63) are consistent if the following expressions for the $\mathcal{G}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$ are used.

$$\begin{aligned} \mathcal{G}_{1,1}'(t_2,t_1,\mathbf{k}) &= G_{1,1}'(t_2,t_1,\mathbf{k})+\int_0^\beta ds_2 G_{1,1}'(t_2,s_2,\mathbf{p})N_{2,0}'(\mathbf{p})\int_0^\beta ds_1 L_{0,2}'(t_1,s_1,\mathbf{p})+L_{2,0}'(t_2,\beta,\mathbf{p})N_{0,2}'(\mathbf{p})G_{1,1}'(\beta,t_1,\mathbf{p}) \\ &\quad +\int_0^\beta ds G_{1,1}'(t_2,s,\mathbf{p})N_{1,1}'(\mathbf{p})G_{1,1}'(\beta,t_1,\mathbf{p})+L_{2,0}'(t_2,\beta,\mathbf{p})N_{1,1}'(-\mathbf{p})\int_0^\beta ds L_{0,2}'(t_1,s,\mathbf{p})R(\beta,-\mathbf{p}), \end{aligned} \quad (93)$$

$$\mathcal{G}_{0,2}'(t_2,t_1,\mathbf{k})=L_{0,2}'(t_2,t_1,\mathbf{k})+2N_{1,1}'(\mathbf{p})G_{1,1}'(\beta,t_2,\mathbf{p})\int_0^\beta dsL_{0,2}'(s,t_1,\mathbf{p})+N_{0,2}'(\mathbf{p})G_{1,1}'(\beta,t_2,\mathbf{p})$$
$$\times\int_0^\beta dsG_{1,1}'(\beta,s,-\mathbf{p})\phi(s,t_1,-\mathbf{p})R^{-1}(\beta,-\mathbf{p})+N_{2,0}'(\mathbf{p})\int_0^\beta ds_2ds_1L_{0,2}'(t_2,s_2,\mathbf{p})L_{0,2}'(s_1,t_1,\mathbf{p}), \quad (94)$$

$$\mathcal{G}_{2,0}'(t_2,t_1,\mathbf{k})=L_{2,0}'(t_2,t_1,\mathbf{k})+2N_{1,1}'(\mathbf{p})\int_0^\beta dsG_{1,1}'(t_2,s,\mathbf{p})L_{2,0}'(\beta,t_1,\mathbf{p})$$
$$+N_{2,0}'(\mathbf{p})\int_0^\beta ds_2ds_1G_{1,1}'(t_2,s_2,\mathbf{p})G_{1,1}'(t_1,s_1,-\mathbf{p})R^{-1}(t_1,-\mathbf{p})+N_{0,2}'(\mathbf{p})L_{2,0}'(t_2,\beta,\mathbf{p})L_{2,0}'(\beta,t_1,\mathbf{p}), \quad (95)$$

where $\mathbf{k}\rightarrow\mathbf{p}$ when $\mathbf{k}$ cannot be zero, and where the function $\phi(t_2,t_1,\mathbf{p})$ in Eq. (94) is given by

$$\phi(t_2,t_1,\mathbf{p})\equiv R(t_2,\mathbf{p})\int_0^\beta dsG_0(t_2,s,\mathbf{p})G_0^{(1)-1}(s,t_1,\mathbf{p}). \quad (96)$$

The determination of the expression for $\mathcal{G}_{1,1}'$, Eq. (93) is completely straightforward. One has to be very careful in the determination of $\mathcal{G}_{0,2}'$ and $\mathcal{G}_{2,0}'$, however. The reason is associated with the convention which we have adopted above Eq. (66). In order to simplify the treatment of these two quantities we have first made the replacement $(t_2,\mathbf{p})\rightleftharpoons(t_1,-\mathbf{p})$ in the $G_{1,1}(t,s,-\mathbf{p})$ terms of Eqs. (67) and (68) in MI. This relabeling can be done because the functions $\mathcal{G}_{\mu,\nu}(t_2,t_1,\mathbf{k})$ only occur as internal lines for which the variables t_2, t_1, and $\mathbf{p}$ are each integrated over their full range of variation. After doing this, one finds that there is one term in both $\mathcal{G}_{0,2}$ and $\mathcal{G}_{2,0}$ which involves two $G_{1,1}$ functions. For each of these terms, one must replace the transformation functions $G_0^{(1)}$ by G_0 in Eqs. (62) and (63), and this explains the appearance of the functions $\phi(s,t_1)$ and $R^{-1}(t_1)$ in the third terms of Eqs. (94) and (95), There seems to be no simpler way of treating the functions $\mathcal{G}_{0,2}$ and $\mathcal{G}_{2,0}$. With this qualifying discussion, Eqs. (61)–(63) have been demonstrated to be consistent with the other equations of the Λ transformation. In Figs. 2–4 we show Eqs. (93)–(95) diagrammatically, using the graphical notation of MI.

We now return to Eqs. (86) and (87) and write down the relations between $K_{0,2}'(\mathbf{p})$ and $K_{2,0}'(\mathbf{p})$ and between $N_{0,2}'(\mathbf{p})$ and $N_{2,0}'(\mathbf{p})$, with the aid of Eqs. (39) in MI.

$$\begin{aligned}K_{2,0}'(\mathbf{p})&=[1+B(\mathbf{p})][1+B^{(1)}(-\mathbf{p})]\zeta^{-1}(\beta,\mathbf{p})\\&\quad\times\exp[\beta(\epsilon_1(-\mathbf{p})-2(g+\Delta^{(0)}))]K_{0,2}'(\mathbf{p}),\\N_{2,0}'(\mathbf{p})&=[1+B(\mathbf{p})][1+B^{(1)}(-\mathbf{p})]\zeta^{-1}(\beta,\mathbf{p})\\&\quad\times\exp[\beta(\epsilon_1(-\mathbf{p})-2(g+\Delta^{(0)}))]N_{0,2}'(\mathbf{p}).\end{aligned} \quad (97)$$

Thus, there are only four independent functions in Eqs. (88)–(90) instead of six. In fact, if one writes down the corresponding identities relating $K_{0,2}'(\mathbf{p})$ and

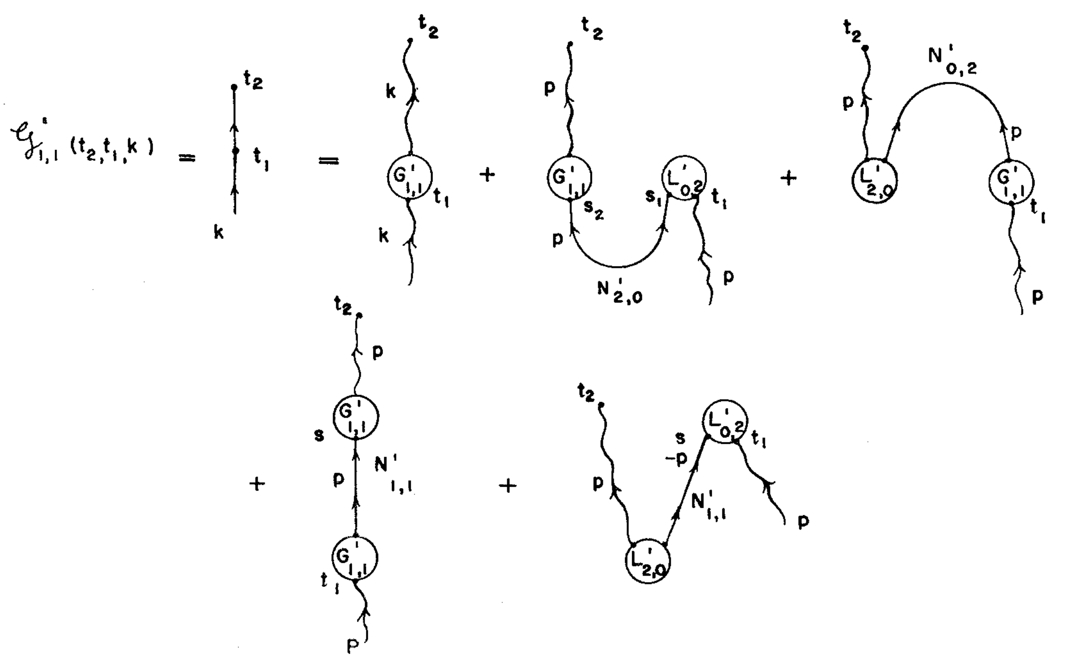

FIG. 2. The graphical representation of Eq. (93) for $\mathcal{G}_{1,1}'(t_2,t_1,\mathbf{k})$, where $\mathbf{k}\rightarrow\mathbf{p}$ when $\mathbf{k}$ cannot be zero. A factor $R(\beta,-\mathbf{p})$ multiplies the last term.

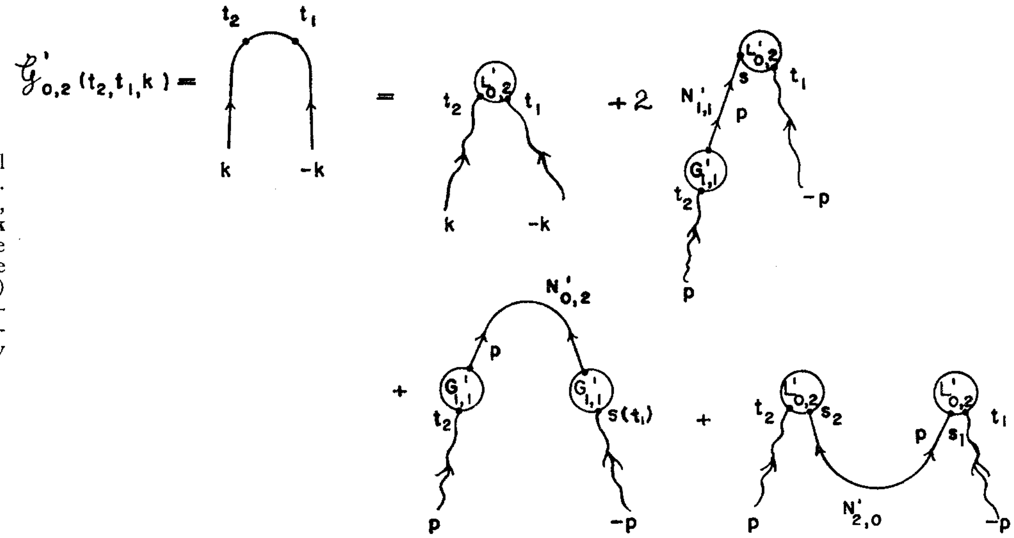

FIG. 3. The graphical representation of Eq. (94) for $\mathcal{G}_{0,2}'(t_2,t_1,\mathbf{k})$, where $\mathbf{k}\rightarrow\mathbf{p}$ when $\mathbf{k}$ cannot be zero. The third term must be multiplied by $\phi(s,t_1,-\mathbf{p})$ $R^{-1}(\beta,-\mathbf{p})$ and an integral over s then performed [indicated by the notation $s(t_1)$].

$K_{2,0}'(-\mathbf{p})$ as well as $N_{0,2}'(-\mathbf{p})$ and $N_{2,0}'(\mathbf{p})$, then one finds that the two Eqs. (88) are identical. The equivalence of the Eqs. (89) and (90) follows from (97). We may now make a partial solution to the coupled integral Eqs. (88) and (89), obtaining the following expressions for $N_{1,1}'(\mathbf{p})$ and $N_{0,2}'(\mathbf{p})$ [simplified because of our introduction of the minus signs in the last term of the second of Eqs. (88)]:

$$N_{1,1}'(\mathbf{p})=\nu'(\mathbf{p})[1-\nu'(-\mathbf{p})K_{1,1}'(-\mathbf{p})][D'(-\mathbf{p})]^{-1}, \quad (98)$$

$$N_{0,2}'(\mathbf{p})=\nu'(\mathbf{p})\nu'(-\mathbf{p})K_{0,2}'(\mathbf{p})R(\beta,-\mathbf{p})[D'(-\mathbf{p})]^{-1}, \quad (99)$$

where

$$D'(\mathbf{p})=[1-\nu'(\mathbf{p})K_{1,1}'(\mathbf{p})][1-\nu'(-\mathbf{p})K_{1,1}'(-\mathbf{p})] -\nu'(\mathbf{p})\nu'(-\mathbf{p})K_{0,2}'(\mathbf{p})K_{0,2}'(\mathbf{p})R(\beta,-\mathbf{p}). \quad (100)$$

We can also use Eqs. (87) in conjunction with Eqs. (40) in MI to determine the limiting forms of $N_{1,1}'(\mathbf{p})$ and $N_{0,2}'(\mathbf{p})$ when $\mathbf{p}\rightarrow 0$. These are [assumed that the factor multiplying $N_{1,1}(\mathbf{p})$ in the first of Eqs. (87) does not vanish when $\mathbf{p}\rightarrow 0$]

$$\lim_{\mathbf{p}\rightarrow 0}[N_{1,1}'(\mathbf{p})]^{-1}=0,$$

$$\lim_{\mathbf{p}\rightarrow 0}\{N_{1,1}'^{-1}(\mathbf{p})N_{0,2}'(\mathbf{p}) \times\exp[-\beta(g+\Delta^{(1)}(\mathbf{p}))][1+B(\mathbf{p})]\}=-1, \quad (101)$$

when $\langle x\rangle>0$.

In an actual application to the calculation of the various functions $K_{\mu,\nu}'(\mathbf{p})$ and $N_{\mu,\nu}'(\mathbf{p})$ for a real or model degenerate Bose system, Eqs. (101) serve as useful checks on the solutions obtained. More useful forms which are completely equivalent to the limits (101) are obtained by combining Eqs. (98)–(100) with

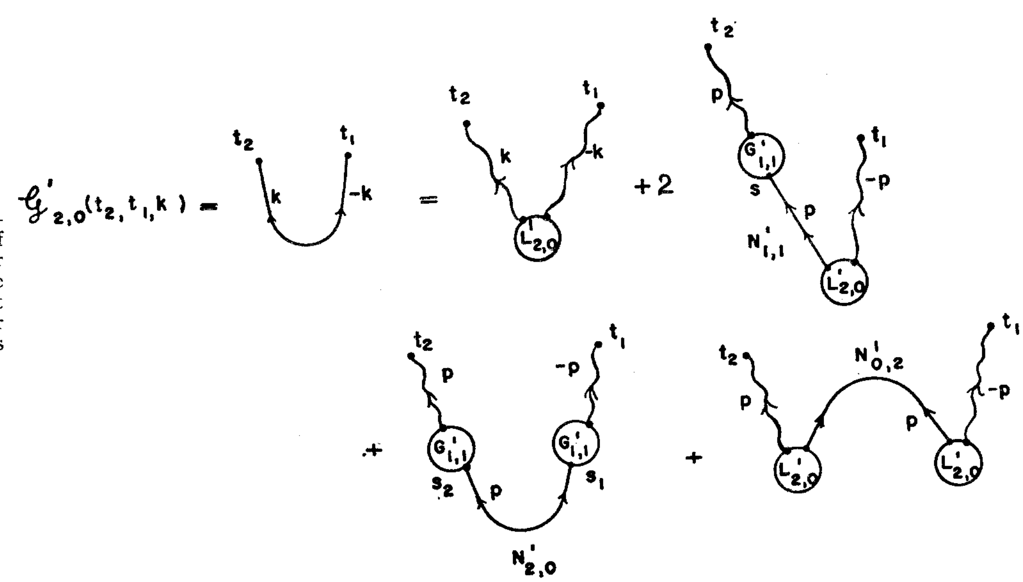

FIG. 4. The graphical representation of Eq. (95) for $\mathcal{G}_{2,0}'(t_2,t_1,\mathbf{k})$, where $\mathbf{k}\rightarrow\mathbf{p}$ when $\mathbf{k}$ cannot be zero. A factor $R^{-1}(t_1,-\mathbf{p})$ multiplies the third term.

(101). This yields the expressions (when $\langle x\rangle>0$)

$$\lim_{\mathbf{p}\to 0}\{[\nu'(\mathbf{p})]^{-1}-K_{1,1}'(\mathbf{p})$$
$$=-[1+B^{(1)}(\mathbf{p})]\zeta^{-1}(\beta,\mathbf{p})\times\exp[-\beta(\Delta^{(0)}+g)]K_{0,2}'(\mathbf{p})\}, \quad (102)$$
$$\lim_{\mathbf{p}\to 0}\{[\nu'(\mathbf{p})]^{-1}-K_{1,1}'(\mathbf{p})$$
$$=-[1+B(\mathbf{p})]^{-1}\exp[\beta(\Delta^{(1)}(\mathbf{p})+g)]K_{2,0}'(\mathbf{p})\},$$

which are equivalent according to the first of Eqs. (97).

As a final matter for this section we write down the relation between the momentum distribution and $N_{1,1}'(\mathbf{p})$. Referring to Eq. (31) in MI and to the first of Eqs. (87) we obtain

$$\langle n(\mathbf{p})\rangle=[1+B(\mathbf{p})]^{-1}\zeta^{-1}(\beta,\mathbf{p})\times\exp[-\beta(g+\Delta^{(0)})]N_{1,1}'(\mathbf{p})-1. \quad (103)$$

The first of the identities (101) then shows that the momentum distribution has a singularity at $\mathbf{p}=0$ when $\langle x\rangle>0$. This is consistent with the interpretation of $\langle x\rangle$ as the macroscopic density of zero-momentum particles. Returning to the discussion at the end of Sec. 5, we see that the function $\zeta(\beta,\mathbf{p})$ has explicitly appeared in the momentum distribution, both in Eq. (103) as well as in the function $\nu'(\mathbf{p})$, Eq. (92), which is an essential quantity in the solution (98) for $N_{1,1}'(\mathbf{p})$. Upon combining Eqs. (98), (92), and (103), we obtain an alternate expression for the momentum distribution:

$$\langle n(\mathbf{p})\rangle=[1+\nu'(\mathbf{p})][1-\nu'(-\mathbf{p})K_{1,1}'(-\mathbf{p})]\times[D'(-\mathbf{p})]^{-1}-1. \quad (104)$$

7. Λ TRANSFORMATION FOR ZERO-MOMENTUM FACTORS

In this section we shall study in detail Eqs. (64) and (65) for the transformation of the zero-momentum factors $G_{\mathrm{out}}(t)$ and $G_{\mathrm{in}}(t)$. This investigation will then complete our study of the consistency of the Λ transformation equations in Sec. 5. We shall begin this study by observing that the function $F(x,\beta,g,\Omega)$ of Eq. (83) in MI is invariant under the Λ transformation. Thus, we may immediately write

$$\Omega F(x,\beta,g,\Omega)=\sum[\text{all different transformed master (0,0) graphs}], \quad (105)$$

where transformed master (0,0) graphs have been defined in Sec. 5. Then we may define transformed functions $\mathcal{K}_{\mathrm{in}}'(t)$ and $\mathcal{K}_{\mathrm{out}}'(t)$ in analogy with Eqs. (93) in MI by the functional differentiations.

$$\mathcal{K}_{\mathrm{in}}'(t)\equiv[(1+B^{(0)})x\Omega\exp\beta(g+\Delta^{(0)})]^{-1}\times\left.\frac{\delta\Omega F}{\delta G_{\mathrm{out}}'(t)}\right|_{\mathcal{G}'},$$
$$\mathcal{K}_{\mathrm{out}}'(t)\equiv[(1+\beta^{(0)})x\Omega\exp\beta(g+\Delta^{(0)})]^{-1}\times\left.\frac{\delta\Omega F}{\delta G_{\mathrm{in}}'(t)}\right|_{\mathcal{G}'}, \quad (106)$$

where, in the first of these expressions, the functional differentiation includes the elimination of one temperature integration. In both of the expressions (106) the internal line factors $\mathcal{G}_{\mu,\nu}'(t_2,t_1,\mathbf{k})$ of the transformed master (0,0) graphs are to be held constant. Also, one difference between the definitions (106) and the expressions (93) in MI is the division by the extra missing line factors $(1+B^{(0)})$ and $\exp(\beta\Delta^{(0)})$, introduced in change (c) for transformed master (μ,ν) graphs (see Sec. 5).

Our task is now to show that a consistent relation exists between the transformation Eqs. (64) and (65) and the definitions (106). In this connection we define functions $K_{\mathrm{out}}'(t)$ and $K_{\mathrm{in}}'(t)$, which are different from the functions of (106) by the equations

$$G_{\mathrm{out}}'(t)\equiv\delta(\beta-t)+K_{\mathrm{out}}'(t),\quad G_{\mathrm{in}}'(t)\equiv1+K_{\mathrm{in}}'(t). \quad (107)$$

Then, from Eqs. (93) in MI and (65) and (106), we have

$$\mathcal{K}_{\mathrm{out}}'(t)=\exp[-(\beta-t)\Delta^{(0)}]K_{\mathrm{out}}(t). \quad (108)$$

But from Eqs. (64) and (33) we also have that

$$K_{\mathrm{out}}'(t)=\exp[-(\beta-t)\Delta^{(0)}]K_{\mathrm{out}}(t)-\int_0^\beta ds G_{\mathrm{out}}'(s)\Lambda^{(0)}(s,t)$$
$$=\mathcal{K}_{\mathrm{out}}'(t)-\int_0^\beta ds G_{\mathrm{out}}'(s)\Lambda^{(0)}(s,t), \quad (109)$$

where

$$\Lambda^{(0)}(t_2,t_1)\equiv\exp[-(t_2-t_1)\Delta^{(0)}][G_0^{(0)}(t_2,t_1)-\delta(t_2-t_1)]$$
$$=[B^{(0)}\delta(t_2-t_1)+(1+B^{(0)})\Delta^{(0)}]\theta(t_2-t_1). \quad (110)$$

We see that Eq. (109) has the form of the Λ transformation Eqs. (59) and (60) or (69) and (70), and that we are permitted to subtract large terms of the form $\int G_0\Lambda^{(0)}$ from $\mathcal{K}_{\mathrm{out}}'(t)$. We shall return to a detailed discussion of Eq. (109) after first examining the Λ transformation of $K_{\mathrm{in}}(t)$.

According to the first of Eqs. (106), Eq. (64), and the definition of transformed master graphs in Sec. 5, the relation which must exist between $\mathcal{K}_{\mathrm{in}}'(t)$ and $K_{\mathrm{in}}(t)$ is

$$[1+B^{(0)}]\mathcal{K}_{\mathrm{in}}'(t)=\exp[-t\Delta^{(0)}]\times\int_0^\beta ds G_0^{(0)}(t,s)K_{\mathrm{in}}(s). \quad (111)$$

Thus, $\mathcal{K}_{\mathrm{in}}'(t)$ differs from $K_{\mathrm{in}}(t)$ by a Λ transformation at the cluster vertex, where the zero-momentum factor is attached. But this last equation can be rewritten with the aid of (110) in the form

$$[1+B^{(0)}]^{-1}\exp[-t\Delta^{(0)}]K_{\mathrm{in}}(t)$$
$$=\mathcal{K}_{\mathrm{in}}'(t)-\int_0^\beta ds\Lambda^{(0)}(t,s)\times\exp[-s\Delta^{(0)}]K_{\mathrm{in}}(s)[1+B^{(0)}]^{-1}.$$

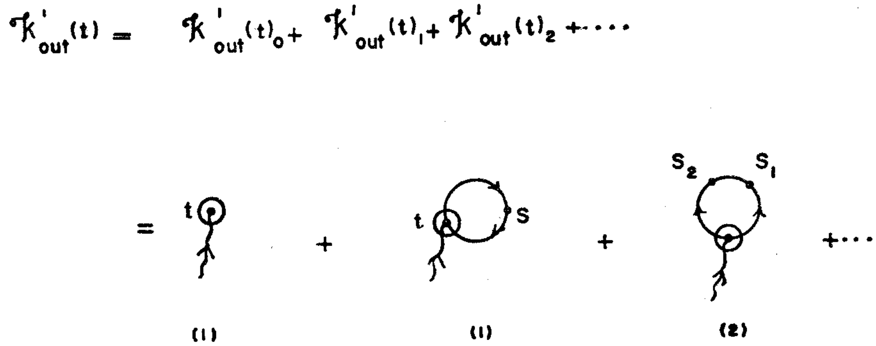

FIG. 5. The graphical expansion of $\mathcal{K}_{out}'(t)$, where only the 3 one-vertex terms have been shown. The external wiggly line in each case is the zero-momentum line which attaches the figure as a zero-momentum factor to a master graph. The number under each graph is its symmetry number [after the differentiation (106)].

The second of Eqs. (107) and Eq. (65) can then be substituted into this equation to give the result

$$K_{in}'(t)=\mathcal{K}_{in}'(t)-\int_0^\beta ds\Lambda^{(0)}(t,s)G_{in}'(s)\,. \quad (112)$$

Equation (112) is seen to be very similar to Eq. (109), and both of them involve the same $\Lambda^{(0)}$ function (110).

With the above analysis of the zero-momentum factors, we have completed the demonstration that the equations of the Λ transformation in Sec. 5 are completely consistent. There remains, however, the clarification of how the function $\Lambda^{(0)}(t_2,t_1)$, Eq. (110), is to be determined in an actual calculation. The answer is that one must always exhibit the functions $\mathcal{K}_{out}'(t)$ and $\mathcal{K}_{in}'(t)$, defined by (106), in the form of Eqs. (109) and (112), respectively. Then the determination of the function $\Lambda^{(0)}(t_2,t_1)$, which must (and will) be the same in both cases is quite straightforward, provided that the difficulties now to be discussed are clearly understood.

In Figs. 5 and 6 we show the graphical expansion of the functions $\mathcal{K}_{out}'(t)$ and $\mathcal{K}_{in}'(t)$, respectively. Only the 3 one-vertex terms have been included in each of these figures, and the graphical notation is that of MI. We observe that after the differentiation (106), one of the zero-momentum factors in ΩF is removed; the corresponding "missing line" is shown in each of the graphs of Figs. 5 and 6. This wiggly line is the one which attaches the corresponding graph (as a zero-momentum factor) to a master graph. Because it is distinguished from the other incoming (outgoing) "missing" line in the first and third graphs of Fig. 5 (6), the factor of $\frac{1}{2}$ in rule (i) in Sec. 6 of MI is not included in these cases.

Consider now the first term in $\mathcal{K}_{out}'(t)$, Fig. 5. The corresponding expression,

$$\mathcal{K}_{out}'(t)_0=\tfrac{1}{2}[(1+B^{(0)})x\Omega\exp\beta(g+\Delta^{(0)})]$$
$$\times\int_0^\beta ds_2ds_1G_{out}'(s_2)G_{out}'(s_1)\,{}^{s_1s_2}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_t' G_{in}'(t)\,, \quad (113)$$

includes two G_{out}' functions and one G_{in}' function. Clearly, only one of the G_{out}' functions can be the one in the second term of (109). Therefore, the contribution of this term to $\Lambda^{(0)}(s,t)$ is determined by setting the selected G_{out}' function equal to $\delta(\beta-s)$. A similar discussion applies to the first term of $\mathcal{K}_{in}'(t)$, Fig. 6, where in this case we must (effectively) set one of the $G_{in}'(t)$ functions equal to $\delta(t'-t)$. The contribution to $\Lambda^{(0)}(s,t)$ will be the same as from (113).

We notice from the above example that the integral equations (109) and (112) are much more nonlinear than Eq. (59), say. Thus, the Λ transformation Eqs. (50)–(60) are only concerned with one $G_{1,1}'$ function along the "**p** line" of an L graph. Similarly, in Eqs. (109) and (112) we *can be* concerned with only one of the zero-momentum factors in any given graph of Figs. 5 or 6, because the Λ transformation, Eqs. (64) and (111), are essentially linear integral transformation equations. But the question then arises as to which zero-momentum

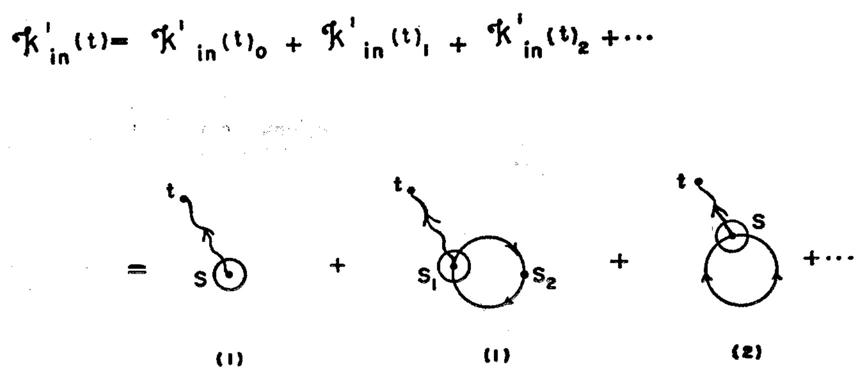

FIG. 6. The graphical expansion of $\mathcal{K}_{in}'(t)$, where only the 3 one-vertex terms have been shown. The external wiggly line in each case is the zero-momentum line which attaches the figure as a zero-momentum factor to a master graph. The number under each graph is its symmetry number [after the differentiation (106)].

factor should be exhibited in (109) or (112) when there are several possibilities. This apparently very complicated situation seems to present no essential difficulties in the application to a Bose system because there always seems to be only one choice which results in the form of the second terms in (109) and (112). [The "two" choices in (113) are really the same.] Therefore, we *assume* that Eqs. (109) and (112) "solve" the zero-momentum self-energy problem and one has only to be careful in their application.

We next consider the third term in $\mathcal{K}_{\text{out}}'(t)$, Fig. 5. The corresponding expression,

$$\mathcal{K}_{\text{out}}'(t)_2=\tfrac{1}{2}(1+B^{(0)})\exp[-\beta(g+\Delta^{(0)})]$$

$$\times\int_0^\beta ds_2ds_1\sum_{\mathbf{k}}\mathcal{G}_{0,2}'(s_2,s_1,\mathbf{k})\,{}^{s_2s_1}\!\begin{bmatrix}\mathbf{k} & -\mathbf{k}\\ 0 & 0\end{bmatrix}_t'G_{\text{in}}'(t)\,, \quad (114)$$

includes *no* G_{out}' functions (explicitly) and only one G_{in}' function. How, then, is Eq. (109) to be used in this case? The answer is that the line-factor $\mathcal{G}_{0,2}'(s_2,s_1,\mathbf{k})$ contains G_{out}' functions (implicitly) and when an expression for this line factor is inserted into (114), then one can identify those terms which are of the form of the second term in (109). A similar analysis applies to the third term of $\mathcal{K}_{\text{out}}'(t)$, Fig. 6. Thus, it seems that Eqs. (109) and (112) can always be applied in actual calculations. In the third paper of this series we shall write down the results of a complete analysis of the terms shown in Figs. 5 and 6 for the case of a dilute gas of Bose hard spheres.

We conclude this section by observing that the factor $\exp\beta(g+\Delta^{(0)})$ occurs frequently in the expressions for quantities after the Λ transformation [see Eqs. (92), (106), and (113)]. This is not just a consequence of the notation which we have introduced, e.g., in Eqs. (28) and (35), because this notation has been introduced only after having studied the model system of a dilute gas of Bose hard spheres. In fact, the only way in which the theory can be made to yield meaningful results is if $g=-\Delta^{(0)}$ in the limit $T\to 0$. Perhaps then, this is a general relation for a degenerate Bose system ($\langle x\rangle>0$). Thus, if we interpret $-\Delta^{(0)}$ as the self-energy of a zero-momentum particle, then the thermodynamic potential per particle, i.e., the "activity," in the system will be given by

$$g=-\Delta^{(0)}\quad\text{when}\quad\langle x\rangle>0\,. \quad (115)$$

[The minus sign can be understood by referring to Eqs. (2) and (3).] We have found no further justification for Eq. (115) other than the mathematical necessity that it must hold in the limit $T\to 0$. On the other hand, it seems to be valid in applications and it can always be independently verified by a thermodynamic calculation of g. We shall assume that (115) is correct, but shall not use this assumption until the following paper.

8. Λ TRANSFORMATION OF GRAND POTENTIAL

In spite of the detailed investigations of the equations of the Λ transformation in the preceding three sections, it is still a nontrivial matter to transform the grand potential. With the aid of Eqs. (61)–(65), (75)–(77) (83), (84), (87), (91), and (111), the expression (91) in MI for the grand potential can be written as

$$\Omega f(x,\beta,g,\Omega)=\tfrac{1}{2}\sum_{\mathbf{p}}\ln\left\{\frac{N_{1,1}'(\mathbf{p})N_{1,1}'(-\mathbf{p})-N_{0,2}'(\mathbf{p})N_{2,0}'(\mathbf{p})R^{-1}(\beta,-\mathbf{p})}{[1+B(\mathbf{p})][1+B(-\mathbf{p})]\zeta(\beta,\mathbf{p})\zeta(\beta,-\mathbf{p})\exp2\beta(g+\Delta^{(0)})}\right\}+[(1+B^{(0)})x\Omega\exp\beta(g+\Delta^{(0)})]$$

$$\times\left[G_{\text{in}}'(\beta)-\int_0^\beta dtG_{\text{out}}'(t)\,\mathcal{K}_{\text{in}}'(t)\right]+\Omega F(x,\beta,g,\Omega)-x\Omega+\sum_{\mathbf{k}}\int_0^\beta dt_1L_{1,1}{}^{(t_1)}(t_1,t_1,\mathbf{k})$$

$$-\tfrac{1}{2}\sum_{\mathbf{k}}\int_0^\beta dt_1dt_2\{\sum_{\substack{(\mu,\nu)\\ \mu+\nu=2}}(1+\delta_{\mu,\nu})\mathcal{G}_{\nu,\mu}'(t_1,t_2,\mathbf{k})\,\mathcal{K}_{\mu,\nu}'(t_2,t_1,\mathbf{k})+K_{2,0}{}^{(1)\prime}(t_1,t_2,\mathbf{k})K_{0,2}{}^{(1)\prime}(t_2,t_1,\mathbf{k})\}\,, \quad (116)$$

where ΩF is given by Eq. (105). Thus, all of the terms except $L_{1,1}{}^{(t_1)}$ are easily transformed by the equations of the Λ transformation. This one term requires special consideration.

According to Eqs. (92) in MI, the difference between $L_{1,1}{}^{(\tau)}(t_2,t_1,\mathbf{k})$ and $L_{1,1}(t_2,t_1,\mathbf{k})$, Eq. (12), is that the upper integration limit in the integral equations for $L_{1,1}{}^{(\tau)}$ is the temperature variable $\tau\neq\beta$. It must be emphasized that this quantity τ is a parameter in the integral equations, and not the variable of integration. Thus, it is only after one has solved (approximately) the integral equations that one may set $\tau=t_1$ for use in (116). However, in the discussion below, one may assume that $\beta>\tau\geqq(t_2,t_1)$.

Now, because the integral equations for $L_{1,1}{}^{(\tau)}(t_2,t_1,\mathbf{k})$ involve the parameter τ instead of β, we must replace β by τ in the relevant Λ transformation equations. Returning to our original examination of Eq. (16), we observe that we must set $B'=B''=0$ in the corresponding expression for $P_0{}^{(\tau)}(t_2,t_1,\mathbf{k})$, because these terms cannot contribute when $\tau<\beta$. Similarly, in the identity (24), we must set $B'=B''=0$, and therefore (24) and (21) together imply that $B^{(<)}(t_2)$ is zero. It also turns out that $C'=C$, when $B'=B''=0$. Finally, we must set $\beta\rightarrow\tau$ in identity (21) [or (24)], and this means that the function $G_0(t_2,t_1)$ of Eq. (19) is changed to a function $G_0^{(\tau)}(t_2,t_1)$, where

$$G_0^{(\tau)}(t_2,t_1)=(1+B)\{[\delta(t_2-t_1)+\Delta_+C_{+,\tau}^{(>)}(t_2)e^{-t_1\Delta+}-\Delta_-C_{-,\tau}^{(>)}(t_2)e^{-t_1\Delta-}]\theta(t_2-t_1)+[\Delta_+C_{+,\tau}^{(<)}(t_2)e^{-t_1\Delta+}-\Delta_-C_{-,\tau}^{(<)}(t_2)e^{-t_1\Delta-}]\theta(t_1-t_2)\}. \quad (117)$$

The implication of this change for the equations of Sec. 4 is that we must replace the $A_i^{(<)}$ and $A_i^{(>)}$ of Eqs. (42) by the quantities

$$\begin{aligned}A_{i,\tau}^{(<)}(t,\mathbf{k})&\equiv[1+B(\mathbf{k})]\exp[-t\epsilon(\mathbf{k})]C_{i,\tau}^{(<)}(t,\mathbf{k}),\\A_{i,\tau}^{(>)}(t,\mathbf{k})&\equiv[1+B(\mathbf{k})]\exp[-t\epsilon(\mathbf{k})]C_{i,\tau}^{(>)}(t,\mathbf{k}),\end{aligned}\quad(118)$$

where $i=+$ or $-$. Similarly, the quantity $\zeta(t,\mathbf{k})$ of (47) is replaced by

$$\zeta_\tau(t,\mathbf{k})=[1+B(\mathbf{k})]^{-1}[A_{+,\tau}^{(>)}(t,\mathbf{k})-A_{-,\tau}^{(>)}(t,\mathbf{k})]. \quad (119)$$

There is no corresponding change in the functions $\Delta_\pm(\mathbf{k})$ of Eqs. (25) and (26), although we have not proved that this is also the case *after* the Λ transformation [see discussion below Eq. (53)].

We can now apply the above discussion to the Λ transformation (Sec. 5) of $L_{1,1}^{(\tau)}(t_2,t_1,\mathbf{k})$. Equations (53) and (51) become in this case

$$G_0^{(\tau)}(t_2,t_1,\mathbf{k})=\delta(t_2-t_1)+e^{t_2\epsilon(\mathbf{k})}\zeta_\tau(t_2,\mathbf{k})\Lambda^{(\tau)}(t_2,t_1,\mathbf{k})\zeta_\tau^{-1}(t_1,\mathbf{k})e^{-t_1\epsilon(\mathbf{k})}, \quad (120)$$

$$\Lambda^{(\tau)}(t_2,t_1,\mathbf{k})=\zeta_\tau^{-1}(t_2)\zeta_\tau(t_1)\{[B\delta(t_2-t_1)+\Delta_+A_{+,\tau}^{(>)}(t_2)e^{t_1\epsilon+}-\Delta_-A_{-,\tau}^{(>)}(t_2)e^{t_1\epsilon-}]\theta(t_2-t_1)+[\Delta_+A_{+,\tau}^{(<)}(t_2)e^{t_1\epsilon+}-\Delta_-A_{-,\tau}^{(<)}(t_2)e^{t_1\epsilon-}]\theta(t_1-t_2)\}. \quad (121)$$

Using these function, $G_0^{(\tau)}$ and $\Lambda^{(\tau)}$, we can then write down the corresponding Λ transformation Eqs. (54)–(60) for the function $L_{1,1}^{(\tau)}(t_2,t_1,\mathbf{k})$ of Eqs. (92) in MI. Thus, one finds that

$$L_{1,1}^{(\tau)}(t_2,t_1)=\zeta_\tau(t_2)e^{t_2\epsilon}\int_0^\tau dsG_{1,1}^{(\tau)\prime}(t_2,s)[P^{(\tau)\prime}(s,t_1)+\Lambda^{(\tau)}(s,t_1)]\zeta_\tau^{-1}(t_1)e^{-t_1\epsilon}, \quad (122)$$

where

$$\begin{aligned}G_{1,1}^{(\tau)\prime}(t_2,t_1,\mathbf{k})&=\delta(t_2-t_1)+L_{1,1}^{(\tau)\prime}(t_2,t_1,\mathbf{k}),\\L_{1,1}^{(\tau)\prime}(t_2,t_1,\mathbf{k})&=\int_0^\tau dsG_{1,1}^{(\tau)\prime}(t_2,s,\mathbf{k})P^{(\tau)\prime}(s,t_1,\mathbf{k}),\\\mathcal{P}^{(\tau)\prime}(t_2,t_1,\mathbf{k})&=[P^{(\tau)\prime}(t_2,t_1,\mathbf{k})+\Lambda^{(\tau)}(t_2,t_1,\mathbf{k})]\\&=\zeta_\tau^{-1}(t_2,\mathbf{k})e^{-t_2\epsilon(\mathbf{k})}\int_0^\tau dsG_0^{(\tau)}(t_2,s,\mathbf{k})P(s,t_1,\mathbf{k})\zeta_\tau(t_1,\mathbf{k})e^{t_1\epsilon(\mathbf{k})}\\&=\mathcal{K}_{1,1}^{(\tau)\prime}(t_2,t_1,\mathbf{k})+\int_0^\tau ds_1ds_2\,\mathcal{K}_{2,0}^{(\tau)\prime}(t_2,s_1,\mathbf{k})G^{(\tau)\prime}(s_2,s_1,-\mathbf{k})\,\mathcal{K}_{0,2}^{(\tau)\prime}(t_1,s_2,\mathbf{k}),\\G^{(\tau)\prime}(t_2,t_1,-\mathbf{k})&=\delta(t_2-t_1)+L^{(\tau)\prime}(t_2,t_1,-\mathbf{k}),\\L^{(\tau)\prime}(t_2,t_1,-\mathbf{k})&=\int_0^\tau dsG^{(\tau)\prime}(t_2,s,-\mathbf{k})K_{1,1}^{(1)\prime}(s,t_1,-\mathbf{k}).\end{aligned}\quad(123)$$

We notice in the last of these equations that the function $K_{1,1}^{(1)\prime}(t_2,t_1,-\mathbf{k})$ of Eq. (70) is *not* changed for the case $\beta\rightarrow\tau$, because the transformation functions $G_0^{(1)}$ and $\Lambda^{(1)}$ of Eqs. (31) and (72), respectively, are independent of β (or τ). On the other hand, the functions $\mathcal{K}_{\mu,\nu}^{(\tau)\prime}(t_2,t_1,\mathbf{k})$ are different from their $\tau\rightarrow\beta$ counterparts, defined by (73). In the present case, the functions $\mathcal{K}_{\mu,\nu}^{(\tau)\prime}(t_2,t_1,\mathbf{k})$ which are related to the unprimed functions by equations similar to (75)–(77), must be defined by the expression

$$\mathcal{K}_{\mu,\nu}^{(\tau)\prime}(t_2,t_1,\mathbf{k})\equiv\sum\text{ [all different transformed master (1,1) } L \text{ graphs, with the external lines transformed by the functions } G_0^{(\tau)}(t_i,s)\text{ and }\zeta^{(\tau)}(t_j)]_{\mathbf{k}}. \quad (124)$$

In this definition (i) and (j) may be either (1) or (2), according to the various cases which can arise for $(\mu+\nu)=2$. It is to be emphasized that the functions $\mathcal{K}_{\mu,\nu}^{(\tau)\prime}(t_2,t_1,\mathbf{k})$ of (124) differ from the corresponding functions of Eq. (73)

only in the transformation of their two external lines. Corresponding internal lines are all transformed by the same functions in (73) and (124).

It is now possible to substitute Eq. (122) into Eq. (116). This yields the completely transformed grand potential

$$\Omega f(x,\beta,g,\Omega)=\tfrac{1}{2}\sum_{\mathbf{p}}\ln\{[1+\langle n(-\mathbf{p})\rangle][1+\nu'(\mathbf{p})][1-\nu'(\mathbf{p})K_{1,1}'(\mathbf{p})]^{-1}\}-x\Omega+[(1+B^{(0)})x\Omega\exp\beta(g+\Delta^{(0)})]$$
$$\times\left[G_{\text{in}}'(\beta)-\int_0^\beta dtG_{\text{out}}'(t)K_{\text{in}}'(t)-\int_0^\beta dt_1dt_2G_{\text{out}}'(t_2)\Lambda^{(0)}(t_2,t_1)G_{\text{in}}'(t_1)\right]$$
$$+\Omega F(x,\beta,g,\Omega)+\sum_{\mathbf{k}}\int_0^\beta dt_1\int_0^{t_1}dt_2G_{1,1}{}^{(t_1)\prime}(t_1,t_2,\mathbf{k})[P^{(t_1)\prime}(t_2,t_1,\mathbf{k})+\Lambda^{(t_1)}(t_2,t_1,\mathbf{k})]$$
$$-\tfrac{1}{2}\sum_{\mathbf{k}}\int_0^\beta dt_1dt_2\{\sum_{\substack{(\mu,\nu)\\ \mu+\nu=2}}(1+\delta_{\mu,\nu})\mathcal{G}_{\nu,\mu}'(t_1,t_2,\mathbf{k})\,\mathcal{K}_{\mu,\nu}'(t_2,t_1,\mathbf{k})+K_{2,0}{}^{(1)\prime}(t_1,t_2,\mathbf{k})K_{0,2}{}^{(1)\prime}(t_2,t_1,\mathbf{k})\}\,,\quad(125)$$

where we have used Eqs. (97)–(100), (103), (104), and (112) to obtain the particular form shown. The sum $\sum_{(\mu,\nu)}$ in the last term is over the three possibilities for which $\mu+\nu=2$. Referring to the discussion of the end of Sec. 5, we see that Λ functions appear explicitly in the grand potential, where it is the grand potential which determines the thermodynamics of a degenerate Bose system. Therefore, particular expressions for the Λ functions directly affect the thermodynamics.

With Eq. (125), we have completed our formal study of the Λ transformation and its effect on the various functions of quantum statistics. The only thing which remains to be done is to exhibit the solutions to the identities (20)–(24), in order to obtain a better feeling for the functions $\Lambda(t_2,t_1,\mathbf{k})$ and $\Lambda^{(\tau)}(t_2,t_1,\mathbf{k})$. This will be done in the following section.

9. DETERMINATION OF THE Λ FUNCTIONS

In this section we discuss the determination of the Λ functions. The zero-momentum Λ function, $\Lambda^{(0)}(t_2,t_1)$, is determined merely by identifying terms of the type (110) when either of the functions $\mathcal{K}_{\text{out}}'(t)$ or $\mathcal{K}_{\text{in}}'(t)$ is explicitly calculated and then cast into the form of Eqs. (109) or (112), respectively. Similarly, the $-\mathbf{k}$ Λ function $\Lambda^{(1)}(t_2,t_1,-\mathbf{k})$ is determined merely by identifying terms of the type (72) when the function $\mathcal{K}_{1,1}{}^{(1)\prime}(t_2,t_1,-\mathbf{k})$ of (78) is written in the form (70).

The functions $\Lambda(t_2,t_1,\mathbf{k})$ and $\Lambda^{(\tau)}(t_2,t_1,\mathbf{k})$ are determined by identifying terms of the type (51) and (121) in the functions $\mathcal{P}'(t_2,t_1,\mathbf{k})$ and $\mathcal{P}^{(\tau)\prime}(t_2,t_1,\mathbf{k})$, respectively [see Eqs. (73), (74), (60), (123), and (124)]. The various quantities $(A_i{}^{(<)},A_i{}^{(>)})$ and $(A_{i,\tau}{}^{(<)},A_{i,\tau}{}^{(>)})$ of Eqs. (51) and (121), respectively, where $i=+$ or $-$, will then turn out to be the solutions of identities of the type (20)–(24).

As was discussed below Eq. (53), only the identities (20) and (21) can be expected to remain invariant under the Λ transformation. The other three identities will, in general, be different after the Λ transformation, from the particular expressions (22)–(24). It is well to observe at this point that the identities of the type (20)–(24) will actually be integral equations which may or may not be solvable by an iteration procedure. Actually, the same is true of the determination of the functions $B^{(0)}$ and $\Delta^{(0)}$ by Eq. (110), and of the functions $B^{(1)}$ and $\Delta^{(1)}$ by Eq. (72). Nevertheless, there is always an algebraic part of the solution to the identities of the type (20)–(24), and it is this part which we shall here be concerned with. The functional dependence on the $(A_i{}^{(<)},A_i{}^{(>)})$ of the various coefficients in these identities must then be dealt with separately.

In the first approximation to a dilute gas of Bose hard spheres with $\langle x\rangle>0$, it is found that the identities (20)–(24) are all unchanged by the Λ transformation. As will be shown in the third paper of this series, one has then only to correctly identify the various coefficients in these equations. It is therefore of value to solve these equations, and we shall now write down the solutions for the particular approximation $B'=B''=B=0$, in which case one finds that $C'=C$ also. This approximation corresponds to the neglect of an excluded volume effect (due to the finite size of the hard cores Ω, is effectively smaller), and this is a very small effect in a dilute gas. In this approximation, one can also omit Eq. (24) and set $B^{(<)}(t_2)=0$ in Eq. (21).

We now insert Eqs. (118) with $B=0$ into the identities (20)–(23), where we need only consider the general case $\tau<\beta$ of Sec. 8, because when $B^{(<)}(t_2)=0$,

$$A_i{}^{(<)}(t_3)=A_{i,\beta}{}^{(<)}(t_2)\,,\text{ and }A_i{}^{(>)}(t_2)=A_{i,\beta}{}^{(>)}(t_2)\,.\quad(126)$$

Thus, we obtain the simpler equations

$$\begin{aligned}
&[A_{+,\tau}{}^{(>)}(t_2)-A_{+,\tau}{}^{(<)}(t_2)]e^{t_2\epsilon_+}\\
&\qquad=1+[A_{-,\tau}{}^{(>)}(t_2)-A_{-,\tau}{}^{(<)}(t_2)]e^{t_2\epsilon_-}\,,\\
&\qquad A_{+,\tau}{}^{(<)}(t_2)e^{\tau\epsilon_+}=A_{-,\tau}{}^{(<)}(t_2)e^{\tau\epsilon_-}\,,\\
&(\Delta_+-D)^{-1}A_{+,\tau}{}^{(>)}(t_2)=(\Delta_--D)^{-1}A_{-,\tau}{}^{(>)}(t_2)\,,\\
&\Delta_+(\Delta_+-D)^{-1}[A_{+,\tau}{}^{(>)}(t_2)-A_{+,\tau}{}^{(<)}(t_2)]e^{t_2\epsilon_+}\\
&\quad=1+\Delta_-(\Delta_--D)^{-1}[A_{-,\tau}{}^{(>)}(t_2)-A_{-,\tau}{}^{(<)}(t_2)]e^{t_2\epsilon_-}\,,
\end{aligned}\quad(127)$$

in which we have used Eqs. (40). The solution to Eqs. (127) is as follows:

$$A_{+,\tau}^{(<)}(t_2)=(\Delta_+-\Delta_-)^{-2}(\Delta_+-D)(\Delta_--D)\times\zeta_\tau(\tau)e^{\tau\epsilon_-}(e^{-t_2\epsilon_+}-e^{-t_2\epsilon_-}),$$
$$A_{-,\tau}^{(<)}(t_2)=(\Delta_+-\Delta_-)^{-2}(\Delta_+-D)(\Delta_--D)\times\zeta_\tau(\tau)e^{\tau\epsilon_+}(e^{-t_2\epsilon_+}-e^{-t_2\epsilon_-}), \quad (128)$$
$$A_{+,\tau}^{(>)}(t_2)=(\Delta_+-\Delta_-)^{-1}(\Delta_+-D)\zeta_\tau(t_2),$$
$$A_{-,\tau}^{(>)}(t_2)=(\Delta_+-\Delta_-)^{-1}(\Delta_--D)\zeta_\tau(t_2),$$

where $\zeta_\tau(t_2)$ is the function of Eq. (119). The explicit expression for this function (when $B=0$) is

$$\zeta_\tau(t_2)=[(\Delta_+-D)e^{\tau\epsilon_+}-(\Delta_--D)e^{\tau\epsilon_-}]^{-1}\times[(\Delta_+-D)e^{(\tau-t_2)\epsilon_+}-(\Delta_--D)e^{(\tau-t_2)\epsilon_-}]$$
$$\xrightarrow[t_2\to\tau]{}[(\Delta_+-D)e^{\tau\epsilon_+}-(\Delta_--D)e^{\tau\epsilon_-}]^{-1}(\Delta_+-\Delta_-). \quad (129)$$

We finally discuss the functions Δ_+ and Δ_- of Eqs. (25) and (26). Equation (25) for the determination of Δ_+ and Δ_- is also found to be valid, to first approximation, for a dilute gas of Bose hard spheres with $\langle x\rangle>0$. The quantity D is given quite generally by the expression

$$D(\mathbf{k})=\epsilon(\mathbf{k})+\epsilon_1(-\mathbf{k})=\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})-\Delta^{(1)}(-\mathbf{k}), \quad (130)$$

where the ϵ's are defined in Eq. (40). The energies $\epsilon_\pm(\mathbf{k})$ are therefore given by

$$\epsilon_\pm=\tfrac{1}{2}[\Delta^{(1)}-A]\pm\tfrac{1}{2}[(D-A)^2-4CD]^{1/2}, \quad (131)$$

according to (26), where the quantities A and C are determined by the derivation of Eq. (25) for the Bose gas of hard spheres. Finally, the quantities (Δ_+-D) and (Δ_--D) in Eqs. (127) and (128) are alternately expressed by

$$[\Delta_\pm(\mathbf{k})-D(\mathbf{k})]=-[\epsilon_\pm(\mathbf{k})+\epsilon_1(-\mathbf{k})]. \quad (132)$$

The assumption which we have made in connection with Eq. (115) that $-\Delta^{(0)}$ is the energy per particle of the zero-momentum "superfluid" in a degenerate Bose system has an implication for the quantities $\epsilon_\pm(\mathbf{k})$, given by Eq. (40) or (131). Because one expects that the elementary excitations in a degenerate Bose system are of a phonon type $\epsilon(\mathbf{k})=Ck$ for low-momentum values, it must be true that the functions $\epsilon_\pm(\mathbf{k})\to O(k)$ as $\mathbf{k}\to 0$. In fact, it will be found in the following paper that this is indeed the case for a dilute gas of Bose hard spheres. What has not yet been clarified is why two functions $\epsilon_+(\mathbf{k})$ and $\epsilon_-(\mathbf{k})$ have appeared in the theory. That such a situation can exist in a degenerate Bose system has previously been suggested by Lieb.[5]

ACKNOWLEDGMENTS

The author is greatly indebted to the people at the Tata Institute of Fundamental Research for the kind hospitality shown to him during his stay there. Also, he wishes to express his appreciation to the United States Educational Foundation in India for the efficient attention which has been given to all of his needs while in India.